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editorial

The Open Secrets of Success of Great Mathematicians

The question that is asked often by many brilliant students and scientists is this-how does one become a great mathematician like Einstein? The answer is simple and an open secret. We try to classify great person as mathematicians, chemists, physicists, philosophers and so on. Great men have no barriers. They are capable of working in many fields, changing the topic according to their thinking at that time and make immense contribution to many subjects. Einstein, de Broglie, Feynman, Dirac etc., the list is very long for enumeration; they are all great thinkers and one cannot classify them.

They all have many common characteristics. We have to closely follow their work in different fields, their way of thinking, particularly cross-thinking of ideas, breaking the custom made barriers, taking any subject as one's own field and hard work to back up. The scientists have a passion for anything they do. Einstein discovered the theory of relativity which was acknowledged by others much later. The Nobel Prize won by Einstein was for experiments on Photoelectricity. Even that contained great ideas about the concept of the photon, its energy and conclusions on light in general. We should not limit our own thinking by taking pride that we are chemists, physicists, mathematicians, historians, etc.

de Broglie studied history, fought in the First World War and after the war, Maurice de Broglie asked him to work in his laboratory. He got his Doctorate in Physics although many examiners were not convinced. His work on Matter waves won him his Nobel Prize. This is one of the corner stores of Modern Physics. With firm conviction, undeterred by criticism; go ahead with your work. You can also join the band of great scientists.

Anil Ahlawat
Editor

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MATHS MUSING

Maths Musing was started in January 2003 issue of Mathematics Today with the suggestion of Shri Mahabir Singh. The aim of Maths Musing is to augment the chances of bright students seeking admission into IITs with additional study material.

During the last 10 years there have been several changes in JEE pattern. To suit these changes Maths Musing also adopted the new pattern by changing the style of problems. Some of the Maths Musing problems have been adapted in JEE benefitting thousand of our readers. It is heartening that we receive solutions of Maths Musing problems from all over India.

Maths Musing has been receiving tremendous response from candidates preparing for JEE and teachers coaching them. We do hope that students will continue to use Maths Musing to boost up their ranks in JEE Main and Advanced.

Prof. Dr. Ramanaiah Gundala, Former Dean of Science and Humanities, Anna University, Chennai

PROBLEM Set 137

JEE MAIN

1. The sum of the last 4 digits of the number 3^{2014} is

(a) 23 (b) 24

(c) 25 (d) 26

2. If the curve $y = x^4 + 4rx + 3s$, $r, s \in R$ lies entirely above the x-axis, then

(a) $r^2 < s^3$ (b) $r^2 > s^3$

(c) $r^4 < s^3$ (d) $r^4 > s^3$

3. If $z = \dfrac{\sqrt{3} - i}{2}$, then $(i^{101} + z^{101})^{103} =$

(a) z (b) $\bar{z}$

(c) iz (d) $i\bar{z}$

4. A 5-digit number is formed using digits 0, 1, 2, 3, 4, 5 without repetition. The probability that it is divisible by 6 is

(a) $\dfrac{7}{50}$ (b) $\dfrac{9}{50}$

(c) $\dfrac{11}{50}$ (d) $\dfrac{13}{50}$

5. Let $\begin{bmatrix} a_{n+1} \\ b_{n+1} \end{bmatrix} = \begin{bmatrix} \sqrt{3}+1 & 1-\sqrt{3} \\ \sqrt{3}-1 & 1+\sqrt{3} \end{bmatrix} \begin{bmatrix} a_n \\ b_n \end{bmatrix}$, $n \in N$.

If $a_1 = b_1 = 1$, then $a_{22} =$

(a) 2^{31} (b) 2^{32}

(c) 2^{33} (d) 3^{33}

JEE ADVANCED

6. In a triangle ABC, if $\tan A = 2\sin 2C$ and $3\cos A = 2 \sin B \sin C$, then $C =$

(a) $\dfrac{\pi}{8}$ (b) $\dfrac{\pi}{6}$

(c) $\dfrac{\pi}{4}$ (d) $\dfrac{\pi}{3}$

COMPREHENSION

In a triangle ABC, the incircle trisect the median through A. If $a = 10$, then

7. Inradius $r =$

(a) $\dfrac{3}{\sqrt{7}}$ (b) $\dfrac{6}{\sqrt{14}}$

(c) $\dfrac{5}{\sqrt{7}}$ (d) $\dfrac{\sqrt{19}}{\sqrt{14}}$

8. The length of the median through A is

(a) $3\sqrt{2}$ (b) $4\sqrt{2}$

(c) $3\sqrt{3}$ (d) $6\sqrt{2}$

INTEGER MATCH

9. The square of the shortest distance between the lines $\vec{r} = s(\hat{i} - \hat{j} - \hat{k})$ and $\vec{r} = 3\hat{j} + t(\hat{i} - \hat{k})$ is

MATCHING LIST

10. Match the following.

List-I		List-II	
P.	$\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{1}{\sqrt{2rn - r^2}}$	1.	$\dfrac{\pi}{6}$
Q.	$\lim\limits_{n\to\infty} \sum\limits_{r=1}^{n} \dfrac{1}{n} \sin \dfrac{r\pi}{2n}$	2.	$\dfrac{\pi}{3}$
R.	$\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \dfrac{1}{\sqrt{4n^2 - r^2}}$	3	$\dfrac{2}{\pi}$
S.	$\lim\limits_{n\to\infty} \sum\limits_{r=0}^{n-1} \dfrac{n}{(n+r)\sqrt{r(2n+r)}}$	4.	$\dfrac{\pi}{2}$

	P	Q	R	S
(a)	3	1	2	4
(b)	1	2	4	3
(c)	2	4	3	1
(d)	4	3	1	2

See Solution set of Maths Musing 136 on page no. 86

Prof. Ramanaiah is the author of MTG JEE(Main & Advanced) Mathematics series

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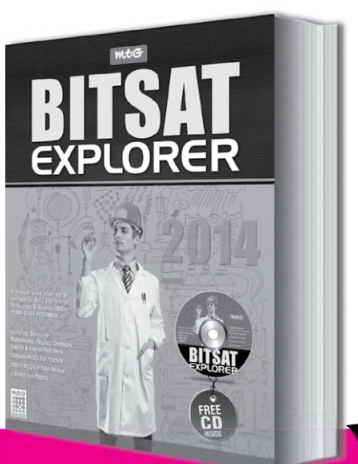

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JEE FINAL TOUCH

Last 3 Years Chapterwise Questions **CLASS-12**

RELATIONS AND FUNCTIONS

1. The function $f : [0, 3] \to [1, 29]$, defined by $f(x) = 2x^3 - 15x^2 + 36x + 1$, is
(a) one-one and onto
(b) onto but not one-one
(c) one-one but not onto
(d) neither one-one nor onto **(2012)**

2. Let $f : (-1, 1) \to R$ be such that $f(\cos 4\theta) = \frac{2}{2 - \sec^2\theta}$ for $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$. Then the values of $f\left(\frac{1}{3}\right)$ is (are)
(a) $1 - \sqrt{\frac{3}{2}}$ (b) $1 + \sqrt{\frac{3}{2}}$
(c) $1 - \sqrt{\frac{2}{3}}$ (d) $1 + \sqrt{\frac{2}{3}}$ **(2012)**

3. Let $f(x) = x^2$ and $g(x) = \sin x$ for all $x \in R$. Then the set of all x satisfying $(fogogof)(x) = (gogof)(x)$, where $(fog)(x) = f(g(x))$, is
(a) $\pm\sqrt{n\pi}, n \in \{0, 1, 2, ...\}$ (b) $\pm\sqrt{n\pi}, n \in \{1, 2, ...\}$
(c) $\frac{\pi}{2} + 2n\pi, n \in \{..., -2, -1, 0, 1, 2, ...\}$
(d) $2n\pi, n \in \{..., -2, -1, 0, 1, 2, ...\}$ **(2011)**

4. Let $f : (0, 1) \to R$ be defined by $f(x) = \frac{b - x}{1 - bx}$, where b is a constant such that $0 < b < 1$. Then
(a) f is not invertible on $(0, 1)$
(b) $f \neq f^{-1}$ on $(0, 1)$ and $f'(b) = \frac{1}{f'(0)}$
(c) $f = f^{-1}$ on $(0, 1)$ and $f'(b) = \frac{1}{f'(0)}$
(d) f^{-1} is differentiable on $(0, 1)$ **(2011)**

INVERSE TRIGONOMETRIC FUNCTIONS

5. The value of $\cot\left\{\sum_{n=1}^{23} \cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right\}$ is

(a) $\frac{23}{25}$ (b) $\frac{25}{23}$ (c) $\frac{23}{24}$ (d) $\frac{24}{23}$ **(2013)**

6. Match list-I with list-II and select the correct answer using the code given below the lists.

	List-I		List-II
(P)	$\left(\frac{1}{y^2}\left(\frac{\cos(\tan^{-1} y) + y\sin(\tan^{-1} y)}{\cot(\sin^{-1} y) + \tan(\sin^{-1} y)}\right)^2 + y^4\right)^{1/2}$ takes value	(1)	$\frac{1}{2}\sqrt{\frac{5}{3}}$
(Q)	If $\cos x + \cos y + \cos z = 0 = \sin x + \sin y + \sin z$ then possible values of $\cos\left(\frac{x - y}{2}\right)$ is	(2)	$\sqrt{2}$
(R)	If $\cos\left(\frac{\pi}{4} - x\right)\cos 2x + \sin x \sin 2x \sec x = \cos x \sin 2x \sec x + \cos\left(\frac{\pi}{4} + x\right)\cos 2x$ then possible value of $\sec x$ is	(3)	$\frac{1}{2}$
(S)	If $\cot(\sin^{-1}\sqrt{1 - x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$, $x \neq 0$ then possible value of x is	(4)	1

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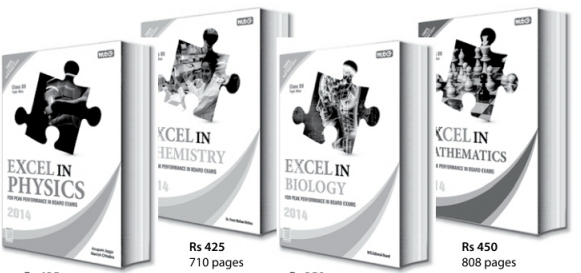

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Codes :

	P	Q	R	S
(a)	4	3	1	2
(b)	4	3	2	1
(c)	3	4	2	1
(d)	3	4	1	2

(2013)

7. Let $f(\theta)=\sin\left(\tan^{-1}\left(\frac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\frac{\pi}{4}<\theta<\frac{\pi}{4}$.

Then the value of $\frac{d}{d(\tan\theta)}(f(\theta))$ is

(Integer Type, 2011)

MATRICES

8. For 3×3 matrices M and N, which of the following statement(s) is (are) **not** correct?

(a) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric.

(b) $MN-NM$ is skew symmetric for all symmetric matrices M and N.

(c) MN is symmetric for all symmetric matrices M and N.

(d) (adj M) (adj N) = adj (MN) for all invertible matrices M and N.

(2013)

9. If P is a 3×3 matrix such that $P^T=2P+I$, where P^T is the transpose of P and I is the 3×3 identity matrix, then there exists a column matrix $X=\begin{bmatrix}x\\y\\z\end{bmatrix}\neq\begin{bmatrix}0\\0\\0\end{bmatrix}$ such that

(a) $PX=\begin{bmatrix}0\\0\\0\end{bmatrix}$ (b) $PX=X$

(c) $PX=2X$ (d) $PX=-X$

(2012)

10. Let M and N be two 3×3 non-singular skew-symmetric matrices such that $MN=NM$. If P^T denotes the transpose of P, then $M^2N^2(M^TN)^{-1}(MN^{-1})^T$ is equal to

(a) M^2 (b) $-N^2$

(c) $-M^2$ (d) MN

(2011)

Paragraph for Question No. 11 to 13

Let a, b and c be three real numbers satisfying

$$[a\ b\ c]\begin{bmatrix}1&9&7\\8&2&7\\7&3&7\end{bmatrix}=\begin{bmatrix}0&0&0\end{bmatrix} \quad \text{.....(E)}$$

11. If the point $P(a, b, c)$, with reference to (E), lies on the plane $2x+y+z=1$, then the value of $7a+b+c$ is

(a) 0 (b) 12 (c) 7 (d) 6

(2011)

12. Let ω be a solution of $x^3-1=0$ with $\text{Im}(\omega)>0$. If $a=2$ with b and c satisfying (E), then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to

(a) –2 (b) 2 (c) 3 (d) –3

(2011)

13. Let $b=6$, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2+bx+c=0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is

(a) 6 (b) 7 (c) 6/7 (d) ∞

(2011)

14. Let $\omega\neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix}1&a&b\\\omega&1&c\\\omega^2&\omega&1\end{bmatrix}$, where each of a, b and c is either ω or ω^2. Then the number of distinct matrices in the set S is

(a) 2 (b) 6 (c) 4 (d) 8

(2011)

15. Let M be a 3×3 matrix satisfying $M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}$, $M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}$, and $M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}$. Then the sum of the diagonal entries of M is

(Integer Type, 2011)

DETERMINANTS

16. Let $P=[a_{ij}]$ be a 3×3 matrix and let $Q=[b_{ij}]$, where $b_{ij}=2^{i+j}a_{ij}$ for $1\le i, j\le 3$. If the determinant of P is 2, then the determinant of the matrix Q is

(a) 2^{10} (b) 2^{11} (c) 2^{12} (d) 2^{13}

(2012)

17. If the adjoint of a 3×3 matrix P is $\begin{bmatrix}1&4&4\\2&1&7\\1&2&3\end{bmatrix}$, then the possible value(s) of the determinant of P is (are)

(a) –2 (b) –1 (c) 1 (d) 2 **(2012)**

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APPLICATION OF DERIVATIVES

18. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is

(a) 6 (b) 4 (c) 2 (d) 0

(2013)

19. A rectangular sheet of fixed perimeter with sides having their lengths in the ratio 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of the removed squares is 100, the resulting box has maximum volume. The lengths of the sides of the rectangular sheet are

(a) 24 (b) 32 (c) 45 (d) 60

(2013)

20. Let $f(x) = x\sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at

(a) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$

(b) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

(c) a unique point in the interval $(n, n+1)$

(d) two points in the interval $(n, n+1)$

(2013)

21. The function $f(x) = 2|x| + |x+2| - ||x+2| - 2|x||$ has a local minimum or a local maximum at $x =$

(a) –2 (b) –2/3 (c) 2 (d) 2/3

(2013)

Paragraph for Q. Nos. 22 and 23

Let $f : [0, 1] \to R$ (the set of all real numbers) be a function. Suppose the function f is twice differentiable, $f(0) = f(1) = 0$ and satisfies $f''(x) - 2f'(x) + f(x) \geq e^x$, $x \in [0, 1]$.

22. Which of the following is true for $0 < x < 1$?

(a) $0 < f(x) < \infty$ (b) $-\frac{1}{2} < f(x) < \frac{1}{2}$

(c) $-\frac{1}{4} < f(x) < 1$ (d) $-\infty < f(x) < 0$

(2013)

23. If the function $e^{-x}f(x)$ assumes its minimum in the interval [0, 1] at $x = 1/4$, which of the following is true?

(a) $f'(x) < f(x)$, $\frac{1}{4} < x < \frac{3}{4}$

(b) $f'(x) > f(x)$, $0 < x < \frac{1}{4}$

(c) $f'(x) < f(x)$, $0 < x < \frac{1}{4}$

(d) $f'(x) < f(x)$, $\frac{3}{4} < x < 1$ **(2013)**

24. Let $p(x)$ be a real polynomial of least degree which has a local maximum at $x = 1$ and a local minimum at $x = 3$. If $p(1) = 6$ and $p(3) = 2$, then $p'(0)$ is

(Integer Type, 2012)

25. Let $f : R \to R$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is

(Integer Type, 2012)

26. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is

(Integer Type, 2011)

INTEGRALS

27. Let $f : \left[\frac{1}{2}, 1\right] \to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right) = 1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

(a) $(2e - 1, 2e)$ (b) $(e - 1, 2e - 1)$

(c) $\left(\frac{e-1}{2}, e-1\right)$ (d) $\left(0, \frac{e-1}{2}\right)$

(2013)

28. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K)

(a) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(b) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(c) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(d) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(2012)

29. The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2 + \ln\frac{\pi + x}{\pi - x}\right)\cos x dx$ is

(a) 0 (b) $\frac{\pi^2}{2} - 4$

(c) $\frac{\pi^2}{2} + 4$ (d) $\frac{\pi^2}{2}$ **(2012)**

Paragraph for Question No. 30 and 31

Let $f(x) = (1 - x)^2\sin^2 x + x^2$ for all $x \in R$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1}\ln t\right) f(t)dt$ for all $x \in (1, \infty)$.

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30. Which of the following is true ?
(a) g is increasing on $(1, \infty)$
(b) g is decreasing on $(1, \infty)$
(c) g is increasing on $(1, 2)$ and decreasing on $(2, \infty)$
(d) g is decreasing on $(1, 2)$ and increasing on $(2, \infty)$
(2012)

31. Consider the statements :
P : There exists some $x \in R$ such that $f(x) + 2x = 2(1 + x^2)$
Q : There exists some $x \in R$ such that $2f(x) + 1 = 2x(1 + x)$. Then
(a) both P and Q are true
(b) P is true and Q is false
(c) P is false and Q is true
(d) both P and Q are false **(2012)**

32. If $f(x) = \int_0^x e^{t^2}(t-2)(t-3)dt$ for all $x \in (0, \infty)$, then
(a) f has a local maximum at $x = 2$
(b) f is decreasing on $(2, 3)$
(c) there exists some $c \in (0, \infty)$ such that $f''(c) = 0$
(d) f has a local minimum at $x = 3$
(2012)

33. The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$ is
(a) $\frac{1}{4}\ln\frac{3}{2}$ (b) $\frac{1}{2}\ln\frac{3}{2}$
(c) $\ln\frac{3}{2}$ (d) $\frac{1}{6}\ln\frac{3}{2}$ **(2011)**

APPLICATION OF INTEGRALS

34. The area enclosed by the curves $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is
(a) $4(\sqrt{2} - 1)$ (b) $2\sqrt{2}(\sqrt{2} - 1)$
(c) $2(\sqrt{2} + 1)$ (d) $2\sqrt{2}(\sqrt{2} + 1)$
(2013)

35. Let S be the area of the region enclosed by $y = e^{-x^2}$, $y = 0$, $x = 0$ and $x = 1$. Then
(a) $S \geq \frac{1}{e}$ (b) $S \geq 1 - \frac{1}{e}$
(c) $S \leq \frac{1}{4}\left(1 + \frac{1}{\sqrt{e}}\right)$
(d) $S \leq \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$ **(2012)**

36. Let the straight line $x = b$ divide the area enclosed by $y = (1 - x)^2$, $y = 0$ and $x = 0$ into two parts $R_1 (0 \leq x \leq b)$ and $R_2 (b \leq x \leq 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals
(a) 3/4 (b) 1/2
(c) 1/3 (d) 1/4 **(2011)**

37. Let $f : [-1, 2] \to [0, \infty)$ be a continuous function such that $f(x) = f(1 - x)$ for all $x \in [-1, 2]$. Let $R_1 = \int_{-1}^{2} xf(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x = -1$, $x = 2$, and the x-axis. Then
(a) $R_1 = 2R_2$ (b) $R_1 = 3R_2$
(c) $2R_1 = R_2$ (d) $3R_1 = R_2$ **(2012)**

DIFFERENTIAL EQUATIONS

38. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x} + \sec\left(\frac{y}{x}\right)$, $x > 0$. Then the equation of the curve is
(a) $\sin\left(\frac{y}{x}\right) = \log x + \frac{1}{2}$ (b) $\operatorname{cosec}\left(\frac{y}{x}\right) = \log x + 2$
(c) $\sec\left(\frac{2y}{x}\right) = \log x + 2$ (d) $\cos\left(\frac{2y}{x}\right) = \log x + \frac{1}{2}$
(2013)

39. If $y(x)$ satisfies the differential equation $y' - y \tan x = 2x \sec x$ and $y(0) = 0$, then
(a) $y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{8\sqrt{2}}$ (b) $y'\left(\frac{\pi}{4}\right) = \frac{\pi^2}{18}$
(c) $y\left(\frac{\pi}{3}\right) = \frac{\pi^2}{9}$ (d) $y'\left(\frac{\pi}{3}\right) = \frac{4\pi}{3} + \frac{2\pi^2}{3\sqrt{3}}$
(2012)

40. Let $f : [1, \infty) \to [2, \infty)$ be a differentiable function such that $f(1) = 2$. If $6\int_1^x f(t)dt = 3xf(x) - x^3$ for all $x \geq 1$, then the value of $f(2)$ is
(Integer Type, 2011)

41. Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\frac{d f(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is
(Integer Type, 2011)

VECTOR ALGEBRA

42. Let $\overrightarrow{PR} = 3\hat{i} + \hat{j} - 2\hat{k}$ and $\overrightarrow{SQ} = \hat{i} - 3\hat{j} - 4\hat{k}$ determine diagonals of a parallelogram $PQRS$ and $\overrightarrow{PT} = \hat{i} + 2\hat{j} + 3\hat{k}$ be another vector. Then the volume of the parallelopiped determined by the vectors $\overrightarrow{PT}$, $\overrightarrow{PQ}$ and $\overrightarrow{PS}$ is

(a) 5 (b) 20 (c) 10 (d) 30

(2013)

43. Consider the set of eight vectors $V = \{a\hat{i} + b\hat{j} + c\hat{k} : a, b, c \in \{-1, 1\}\}$. Three non-coplanar vectors can be chosen from V in 2^p ways. Then p is

(Integer Type, 2013)

44. Two lines $L_1 : x = 5, \frac{y}{3-\alpha} = \frac{z}{-2}$ and $L_2 : x = \alpha, \frac{y}{-1} = \frac{z}{2-\alpha}$ are coplanar. Then α can take value(s)

(a) 1 (b) 2 (c) 3 (d) 4

(2013)

45. Match List-I with List-II and select the correct answer using the code given below the lists:

List-I		List-II	
(P)	Volume of parallelopiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 2. Then the volume of the parallelopiped determined by vectors $2(\vec{a} \times \vec{b})$, $3(\vec{b} \times \vec{c})$ and $(\vec{c} \times \vec{a})$ is	(1)	100
(Q)	Volume of parallelopiped determined by vectors $\vec{a}, \vec{b}$ and $\vec{c}$ is 5. Then the volume of the parallelopiped determined by vectors $3(\vec{a} + \vec{b})$, $(\vec{b} + \vec{c})$, $2(\vec{c} + \vec{a})$ is	(2)	30
(R)	Area of the triangle with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 20. Then the area of the triangle with adjacent sides determined by vectors $(2\vec{a} + 3\vec{b})$ and $(\vec{a} - \vec{b})$ is	(3)	24
(S)	Area of a parallelogram with adjacent sides determined by vectors $\vec{a}$ and $\vec{b}$ is 30. Then the area of the parallelogram with adjacent sides determined by vectors $(\vec{a} + \vec{b})$ and $\vec{a}$ is	(4)	60

Codes :

	P	Q	R	S
(a)	4	2	3	1
(b)	2	3	1	4
(c)	3	4	1	2
(d)	1	4	3	2

(2013)

46. If $\vec{a}, \vec{b}$ and $\vec{c}$ are unit vectors satisfying $|\vec{a} - \vec{b}|^2 + |\vec{b} - \vec{c}|^2 + |\vec{c} - \vec{a}|^2 = 9$, then $|2\vec{a} + 5\vec{b} + 5\vec{c}|$ is

(Integer Type, 2012)

47. If $\vec{a}$ and $\vec{b}$ are vectors such that $|\vec{a} + \vec{b}| = \sqrt{29}$ and $\vec{a} \times (2\hat{i} + 3\hat{j} + 4\hat{k}) = (2\hat{i} + 3\hat{j} + 4\hat{k}) \times \vec{b}$, then a possible value of $(\vec{a} + \vec{b})(-7\hat{i} + 2\hat{j} + 3\hat{k})$ is

(a) 0 (b) 3 (c) 4 (d) 8

(2012)

48. The vector(s) which is/are coplanar with vectors $\hat{i} + \hat{j} + 2\hat{k}$ and $\hat{i} + 2\hat{j} + \hat{k}$, and perpendicular to the vector $\hat{i} + \hat{j} + \hat{k}$ is/are

(a) $\hat{j} - \hat{k}$ (b) $-\hat{i} + \hat{j}$

(c) $\hat{i} - \hat{j}$ (d) $-\hat{j} + \hat{k}$ **(2011)**

THREE DIMENSIONAL GEOMETRY

49. Perpendiculars are drawn from points on the line $\frac{x+2}{2} = \frac{y+1}{-1} = \frac{z}{3}$ to the plane $x + y + z = 3$. The feet of perpendiculars lie on the line

(a) $\frac{x}{5} = \frac{y-1}{8} = \frac{z-2}{-13}$ (b) $\frac{x}{2} = \frac{y-1}{3} = \frac{z-2}{-5}$

(c) $\frac{x}{4} = \frac{y-1}{3} = \frac{z-2}{-7}$ (d) $\frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$

(2013)

50. A line l passing through the origin is perpendicular to the lines

$l_1 : (3+t)\hat{i} + (-1+2t)\hat{j} + (4+2t)\hat{k}, \ -\infty < t < \infty$

$l_2 : (3+2s)\hat{i} + (3+2s)\hat{j} + (2+s)\hat{k}, \ -\infty < s < \infty$

Then the coordinate(s) of the point(s) on l_2 at a distance of $\sqrt{17}$ from the point of intersection of l and l_1 is(are)

(a) $\left(\frac{7}{3}, \frac{7}{3}, \frac{5}{3}\right)$ (b) $(-1, -1, 0)$

(c) $(1, 1, 1)$ (d) $\left(\frac{7}{9}, \frac{7}{9}, \frac{8}{9}\right)$ **(2013)**

51. Consider the lines $L_1 : \frac{x-1}{2} = \frac{y}{-1} = \frac{z+3}{1}$, $L_2 : \frac{x-4}{1} = \frac{y+3}{1} = \frac{z+3}{2}$ and the planes

$P_1 : 7x + y + 2z = 3$, $P_2 : 3x + 5y - 6z = 4$. Let $ax + by + cz = d$ the equation of the plane passing through the point of intersection of lines L_1 and L_2 and perpendicular to planes P_1 and P_2.

Match List-I with List-II and select the correct answer using the code given below the lists:

List-I		List-II	
(P)	a =	(1)	13
(Q)	b =	(2)	–3
(R)	c =	(3)	1
(S)	d =	(4)	–2

Codes :

	P	Q	R	S
(a)	3	2	4	1
(b)	1	3	4	2
(c)	3	2	1	4
(d)	2	4	1	3

(2013)

52. The point P is the intersection of the straight line joining the points $Q(2, 3, 5)$ and $R(1, -1, 4)$ with the plane $5x - 4y - z = 1$. If S is the foot of the perpendicular drawn from the point $T(2, 1, 4)$ to QR, then the length of the line segment PS is

(a) $\frac{1}{\sqrt{2}}$ (b) $\sqrt{2}$ (c) 2 (d) $2\sqrt{2}$

(2012)

53. The equation of a plane passing through the line of intersection of the planes $x + 2y + 3z = 2$ and $x - y + z = 3$ and at a distance $\frac{2}{\sqrt{3}}$ from the point $(3, 1, -1)$ is

(a) $5x - 11y + z = 17$ (b) $\sqrt{2}x + y = 3\sqrt{2} - 1$

(c) $x + y + z = \sqrt{3}$ (d) $x - \sqrt{2}y = 1 - \sqrt{2}$

(2012)

54. If the straight line $\frac{x-1}{2} = \frac{y+1}{k} = \frac{z}{2}$ and $\frac{x+1}{5} = \frac{y+1}{2} = \frac{z}{k}$ are coplanar, then the plane(s) containing these two lines is (are)

(a) $y + 2z = -1$ (b) $y + z = -1$

(c) $y - z = -1$ (d) $y - 2z = -1$ **(2012)**

55. Let $\vec{a} = \hat{i} + \hat{j} + \hat{k}$, $\vec{b} = \hat{i} - \hat{j} + \hat{k}$ and $\vec{c} = \hat{i} - \hat{j} - \hat{k}$ be three vectors. A vector $\vec{v}$ in the plane of $\vec{a}$ and $\vec{b}$, whose projection on $\vec{c}$ is $\frac{1}{\sqrt{3}}$, is given by

(a) $\hat{i} - 3\hat{j} + 3\hat{k}$ (b) $-3\hat{i} - 3\hat{j} - \hat{k}$

(c) $3\hat{i} - \hat{j} + 3\hat{k}$ (d) $\hat{i} + 3\hat{j} - 3\hat{k}$ **(2011)**

56. Let $\vec{a} = -\hat{i} - \hat{k}$, $\vec{b} = -\hat{i} + \hat{j}$ and $\vec{c} = \hat{i} + 2\hat{j} + 3\hat{k}$ be three given vectors. If $\vec{r}$ is a vector such that $\vec{r} \times \vec{b} = \vec{c} \times \vec{b}$ and $\vec{r} \cdot \vec{a} = 0$, then the value of $\vec{r} \cdot \vec{b}$ is

(Integer Type, 2011)

PROBABILITY

57. Four persons independently solve a certain problem correctly with probabilities $\frac{1}{2}, \frac{3}{4}, \frac{1}{4}, \frac{1}{8}$. Then the probability that the problem is solved correctly by at least one of them is

(a) $\frac{235}{256}$ (b) $\frac{21}{256}$

(c) $\frac{3}{256}$ (d) $\frac{253}{256}$ **(2013)**

58. Of the three independent events, E_1, E_2 and E_3 the probability that only E_1 occurs is α, only E_2 occurs is β and only E_3 occurs is γ. Let the probability p that none of the events E_1, E_2 or E_3 occurs satisfy the equations $(\alpha - 2\beta)p = \alpha\beta$ and $(\beta - 3\gamma)p = 2\beta\gamma$. All the given probabilities are assumed to lie in the interval (0, 1). Then

$$\frac{\text{Probability of occurrence of } E_1}{\text{Probability of occurrence of } E_3} =$$

(Integer Type, 2013)

Paragraph for Q. Nos. 59 and 60

A box B_1 contains 1 white ball, 3 red balls and 2 black balls. Another box B_2 contains 2 white balls, 3 red balls and 4 black balls. A third box B_3 contains 3 white balls, 4 red balls and 5 black balls.

59. If 1 ball is drawn from each of the boxes B_1, B_2 and B_3, the probability that all 3 drawn balls are of the same colour is

(a) $\frac{82}{648}$ (b) $\frac{90}{648}$ (c) $\frac{558}{648}$ (d) $\frac{566}{648}$

(2013)

60. If 2 balls are drawn (without replacement) from a randomly selected box and one of the balls is white and the other ball is red, the probability that these 2 balls are drawn from box B_2 is

(a) $\frac{116}{181}$ (b) $\frac{126}{181}$

(c) $\frac{65}{181}$ (d) $\frac{55}{181}$ **(2013)**

61. A ship is fitted with three engines E_1, E_2 and E_3. The engines function independently of each other with respective probabilities $\frac{1}{2}, \frac{1}{4}$ and $\frac{1}{4}$. For the ship to be operational at least two of its engines must function. Let X denote the event that the ship is operational and let X_1, X_2 and X_3 denote respectively the events that the engines E_1, E_2 and E_3 are functioning. Which of the following is (are) true?

(a) $P\left[X_1^c \mid X\right] = \frac{3}{16}$

(b) P [Exactly two engines of the ship are functioning $\mid X] = \frac{7}{8}$

(c) $P\left[X \mid X_2\right] = \frac{5}{16}$

(d) $P'\left[X \mid X_1\right] = \frac{7}{16}$ **(2012)**

62. Let X and Y be two events such that $P(X \mid Y) = \frac{1}{2}$, $P(Y \mid X) = \frac{1}{3}$ and $P(X \cap Y) = \frac{1}{6}$. Which of the following is(are) correct?

(a) $P(X \cup Y) = \frac{2}{3}$

(b) X and Y are independent

(c) X and Y are not independent

(d) $P(X^c \cap Y) = \frac{1}{3}$ **(2012)**

Paragraph for Question No. 63 and 64

Let U_1 and U_2 be two urns such that U_1 contains 3 white and 2 red balls, and U_2 contains only 1 white ball. A fair coin is tossed. If head appears then 1 ball is drawn at random from U_1 and put into U_2. However, if tail appears then 2 balls are drawn at random from U_1 and put into U_2. Now 1 ball is drawn at random from U_2.

63. The probability of the drawn ball from U_2 being white is

(a) $\frac{13}{30}$ (b) $\frac{23}{30}$ (c) $\frac{19}{30}$ (d) $\frac{11}{30}$ **(2011)**

64. Given that the drawn ball from U_2 is white, the probability that head appeared on the coin is

(a) $\frac{17}{23}$ (b) $\frac{11}{23}$

(c) $\frac{15}{23}$ (d) $\frac{12}{23}$ **(2011)**

65. Let E and F be two independent events. The probability that exactly one of them occurs is $\frac{11}{25}$ and the probability of none of them occurring is $\frac{2}{25}$. If $P(T)$ denotes the probability of occurrence of the event T, then

(a) $P(E) = \frac{4}{5}, P(F) = \frac{3}{5}$ (b) $P(E) = \frac{1}{5}, P(F) = \frac{2}{5}$

(c) $P(E) = \frac{2}{5}, P(F) = \frac{1}{5}$ (d) $P(E) = \frac{3}{5}, P(F) = \frac{4}{5}$ **(2011)**

MISCELLANEOUS

66. Match the statements given in column I with the values given in column II.

Column I		Column II	
(A)	If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}, \vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p)	$\frac{\pi}{6}$
(B)	If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q)	$\frac{2\pi}{3}$
(C)	The value of $\frac{\pi^2}{\ln 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r)	$\frac{\pi}{3}$
(D)	The maximum value of $\left\lvert \text{Arg}\left(\frac{1}{1-z}\right)\right\rvert$ for $\lvert z\rvert = 1, z \neq 1$ is given by	(s)	π
		(t)	$\frac{\pi}{2}$

(2011)

67. Match the statements given in column I with the intervals/union of intervals given in column II.

Column I		Column II	
(A)	The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number, } \lvert z\rvert = 1, z \neq \pm 1\right\}$ is	(p)	$(-\infty, -1) \cup (1, \infty)$
(B)	The domain of the function $f(x) = \sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q)	$(-\infty, 0) \cup (0, \infty)$
(C)	If $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r)	$[2, \infty)$
(D)	If $f(x) = x^{3/2}(3x - 10)$, $x \ge 0$, then $f(x)$ is increasing in	(s)	$(-\infty, -1] \cup [1, \infty)$
		(t)	$(-\infty, 0] \cup [2, \infty)$

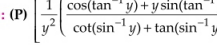
(2011)

SOLUTIONS

1. (b) : $f(x) = 2x^3 - 15x^2 + 36x + 1$

We have $f'(x) = 6x^2 - 30x + 36 = 6(x^2 - 5x + 6)$

$= 6(x-1)(x-5)$



$f(0) = 1$

$f(3) = 54 - 135 + 1 - 8 + 1 = 28$

$f(2) = 16 - 60 + 72 + 1 = 29$

Then the range = (1, 29)

Hence the given function is ONTO. But it is NOT one-one, as f' takes both positive and negative values.

2. (a, b) : $f(\cos 4\theta) = \dfrac{2}{2 - \sec^2\theta}$

We have let $\cos 4\theta = t$

$\Rightarrow \quad 2\cos^2 2\theta - 1 = t \;\Rightarrow\; \cos^2 2\theta = \dfrac{1+t}{2}$

For $t = \dfrac{1}{3}$ we have $\cos^2 2\theta = \dfrac{2}{3}$

$\cos 2\theta = \sqrt{\dfrac{2}{3}}$ or $\cos 2\theta = -\sqrt{\dfrac{2}{3}}$

$$f(\cos 4\theta) = \frac{2}{2 - \dfrac{1}{\cos^2\theta}} = \frac{2\cos^2\theta}{2\cos^2\theta - 1} = \frac{1 + \cos 2\theta}{\cos 2\theta} = 1 + \frac{1}{\cos 2\theta}$$

Hence, $f\left(\dfrac{1}{3}\right) = 1 + \dfrac{\sqrt{3}}{2}$ or $1 - \dfrac{\sqrt{3}}{2}$

3. (a) : $f(x) = x^2$, $g(x) = \sin x$

$gof(x) = \sin x^2$, $gogof(x) = \sin(\sin x^2)$

Again $fogogof(x) = (\sin(\sin x^2))^2 = \sin^2(\sin x^2)$

Now $\sin^2(\sin x^2) = \sin(\sin x^2)$

$\Rightarrow \quad \sin(\sin x^2) = 0$ or 1

$\sin x^2 = n\pi,\ 2n\pi + \pi/2,\ n \in z,\ \sin x^2 = 0$

$\Rightarrow \quad x^2 = n\pi \quad \therefore \quad x = \sqrt{n\pi},\ n \geq 0,\ n \in z.$

4. (a) : $f(x) = \dfrac{x-b}{bx-1}$

As the range of f is not R (co-domain), hence the function is not invertible.

5. (b): $\cot\left(\sum_{n=1}^{23}\left(\cot^{-1}\left(1 + \sum_{k=1}^{n} 2k\right)\right)\right)$

$$= \cot\sum_{n=1}^{23}\cot^{-1}\left(1 + \frac{2(n+1)n}{2}\right)$$

$$= \cot\sum_{n=1}^{23}\cot^{-1}(1 + n(n+1))$$

$$= \cot\sum_{n=1}^{23}\tan^{-1}\left(\frac{1}{1+n(n+1)}\right) = \cot(\tan^{-1}24 - \tan^{-1}1)$$

$$= \cot\left(\tan^{-1}\left(\frac{24-1}{1+24}\right)\right) = \cot\left(\tan^{-1}\frac{23}{25}\right)$$

$$= \cot\left(\cot^{-1}\frac{25}{23}\right) = \frac{25}{23}$$

6. (b): **(P)** $\left[\dfrac{1}{y^2}\left(\dfrac{\cos(\tan^{-1}y) + y\sin(\tan^{-1}y)}{\cot(\sin^{-1}y) + \tan(\sin^{-1}y)}\right)^2 + y^4\right]^{1/2}$

$$= \frac{\left\{\dfrac{1}{\sqrt{1+y^2}} + \dfrac{y^2}{\sqrt{1+y^2}}\right\}^2}{y^2\left(\dfrac{\sqrt{1-y^2}}{y} + \dfrac{y}{\sqrt{1-y^2}}\right)^2} + y^4$$

$$= \left[\frac{1}{y^2}\cdot y^2(1-y^4) + y^4\right]^{1/2} = 1$$

(Q) $\cos x + \cos y = -\cos z$ and $\sin x + \sin y = -\sin z$

Squaring and adding we get $2 + 2\cos(x-y) = 1$

$\therefore \quad \cos(x-y) = -1/2$

$$2\cos^2\left(\frac{x-y}{2}\right) - 1 = -\frac{1}{2}. \quad \therefore \quad \cos\left(\frac{x-y}{2}\right) = \frac{1}{2}$$

(R) $\cos 2x\left\{\cos\left(\dfrac{\pi}{4} - x\right) - \cos\left(\dfrac{\pi}{4} + x\right)\right\} + 2\sin^2 x$

$$= 2\sin x\cos x$$

$= \cos 2x\,(\sqrt{2}\sin x) + 2\sin^2 x = 2\sin x\cos x$

$= \sqrt{2}\sin x\,[\cos 2x + \sqrt{2}\sin x - \sqrt{2}\cos x] = 0$

$\Rightarrow$ Either $\sin x = 0$ or $\cos^2 x - \sin^2 x = \sqrt{2}(\cos x - \sin x)$

$\Rightarrow \quad \sec x = 1$ or $\cos x = \sin x$

$\Rightarrow \quad \sec x = 1$ or $\sec x = \sqrt{2}$

(S) $\cot(\sin^{-1}\sqrt{1-x^2}) = \sin(\tan^{-1}(x\sqrt{6}))$

$$\Rightarrow \quad \frac{|x|}{\sqrt{1-x^2}} = \frac{\sqrt{6}x}{\sqrt{1+6x^2}}, \quad x > 0$$

$\Rightarrow \quad 1 + 6x^2 = 6 - 6x^2 \;\Rightarrow\; 12x^2 = 5 \quad \therefore \quad x = \dfrac{\sqrt{5}}{2\sqrt{3}}$

7. (1) : $\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \sin^{-1}\left(\dfrac{\sin\theta}{\sqrt{\sin^2\theta + \cos 2\theta}}\right)$

$= \sin^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos^2\theta}}\right) = \sin^{-1}\left(\dfrac{\sin\theta}{\cos\theta}\right)$ as $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$

Thus $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos^2\theta}}\right)\right) = \sin\left(\sin^{-1}\left(\dfrac{\sin\theta}{\cos\theta}\right)\right)$

$= \sin(\sin^{-1}(\tan\theta)) = \tan\theta$

8. (c, d) :

$$(N^TMN)^T = \begin{cases} N^TM^TN = N^TMN, \text{ if } M \text{ is symmetric} \\ -N^TMN, \text{ if } M \text{ is skew symmetric} \end{cases}$$

Again $(MN - NM)^T = (MN)^T - (NM)^T$

$= N^TM^T - M^TN^T = NM - MN = -(MN - NM)$

$(MN)^T = N^TM^T = NM \neq MN$ (in general)

adj(MN) = (adj N)(adj M)

and they don't commute in general.

Thus (a) and (b) are correct and (c) and (d) are incorrect.

9. (d) : We have $P^T = 2P + I$

We get $P^T - 2P = I$ .....(i)

Taking transpose, we have $(P^T - 2P)^T = I^T$

$\Rightarrow \quad P - 2P^T = I$ ..... (ii)

From (i) and (ii) on eliminating P^T we have

$-4P + P = 3I \Rightarrow P = -I \quad \therefore \quad P + I = 0$

Thus $(P + 1)X = 0 \quad \Rightarrow \quad PX = -X$

10. (c) : THE QUESTION IS WRONG, AS GIVEN. AN ODD ORDER SKEW SYMMETRIC MATRIX CAN'T BE INVERTIBLE. HAD THE MATRIX BE OF EVEN ORDER, IT COULD HAVE BEEN CORRECT.

Assuming M, N to be even order, we have

$M^2N^2(M^TN)^{-1}(MN^{-1})^T = M^2N^2(-MN)^{-1}(MN^{-1})^T$

$= -M^2N^2(MN)^{-1}(MN^{-1})^T$

$= -M^2N^2(N^{-1}M^{-1})((N^{-1})^TM^T)$

$= -M^2N^2(N^{-1}M^{-1})(N^T)^{-1}(M^T)$

$= -M^2N^2(N^{-1}M^{-1})(-N^{-1}(-M))$

$= -M^2N^2N^{-1}M^{-1}N^{-1}M$

$= -M^2(NM^{-1})(N^{-1}M) = -M^2(NM^{-1})(MN^{-1})$

$= -M^2NIN^{-1} = -M^2NN^{-1} = -M^2$

11-13 :

We have $\left.\begin{matrix} a + 8b + 7c = 0 \\ 9a + 2b + 3c = 0 \\ a + b + c = 0 \end{matrix}\right\}$

Let $\Delta = \begin{vmatrix} 1 & 8 & 7 \\ 9 & 2 & 3 \\ 1 & 1 & 1 \end{vmatrix} = 1(-1) - 8(6) + 7(7) = 0$

Let $c = k$ then a and b can be expressed in terms of k.

$\left.\begin{matrix} a + b = -k \\ a + 8b = -7k \end{matrix}\right\}$ gives $a = -\frac{1}{7}k$ and $b = -\frac{6}{7}k$

Thus $(a, b, c) = k\left(-\frac{1}{7}, -\frac{6}{7}, 1\right)$

11. (a) : $2a + b + c = 1$, we have

$-\frac{2k}{7} - \frac{6k}{7} + k = 1 \Rightarrow k = -7$

Thus $7a + b + c = 7 + 6 - 7 = 6$.

12. (a) : For $a = 2$, k turns to be $k = -14$

$\therefore \quad b = 12, \; c = -14$

$\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + 3\omega^{14}$

$= 3\omega + 1 + 3\omega^2 = 3(\omega + \omega^2) + 1 = -3 + 1 = -2.$

13. (b) : $b = 6 \Rightarrow k = -7$

$\therefore \quad a = 1$ and $c = -7$

So the equation become $x^2 + 6x - 7 = 0 \; \therefore \; x = -7, 1$

$\sum_{n=0}^{\infty}\left(\frac{1}{\alpha} + \frac{1}{\beta}\right)^n = \sum_{n=0}^{\infty}\left(\frac{6}{7}\right)^n = \frac{1}{1 - (6/7)} = 7.$

14. (a) : Given $a, b, c \in \{\omega, \omega^2\}$

Let $M = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ but $M = 1 - (a + c)\omega + ac\omega^2$

observe the table

a	c	$a + c$	ac
ω	ω^2	-1	1
ω	ω	2ω	ω^2
ω^2	ω	-1	1
ω^2	ω^2	$2\omega^2$	ω

The possible value of $1 - (a + c)\omega + ac\omega^2$ are

$1 + \omega + \omega^2 = 0, \; 1 - 2\omega^2 + \omega = -3\omega^2, \; 1 + \omega + \omega^2 = 0,$

$1 - 2\omega^3 + \omega^3 = 0$

Thus the det $A \neq 0$ if and only if $a = c = \omega$

Thus b can take two values $b = \omega, \omega^2$.

Hence the number of matrices is 2.

15. (9) : Let $M = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & 2 \end{bmatrix}$

$\left.\begin{matrix} a = 0 \\ e = 2 \\ g = 7 \end{matrix}\right\}$ by solving simultaneous equation.

from the system of equations we get

Sum of diagonal entries $= 0 + 2 + 7 = 9$.

16. (d) : Let $Q = \begin{vmatrix} 2^2a_{11} & 2^3a_{12} & 2^4a_{13} \\ 2^3a_{21} & 2^4a_{21} & 2^5a_{23} \\ 2^4a_{31} & 2^5a_{32} & 2^6a_{33} \end{vmatrix}$

$= 2^9\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2a_{21} & 2a_{22} & 2a_{23} \\ 2^2a_{31} & 2^2a_{32} & 2^2a_{33} \end{vmatrix}$

$= 2^9 \cdot 2 \cdot 2^2\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

$= 2^{12}\det P = 2^{12} \cdot 2 = 2^{13}$

17. (a, d) : det(adj P) = (det P)^2

Recall that det(adj P) = (det P)$^{n-1}$ for $n \times n$ matrix.

As det(adj P) = 1(3 – 7) – 4(6 – 7) + 4(2 – 1)

= – 4 + 4 + 4 = 4.

Hence det P = 2 or –2.

18. (c) : 1st solution : $f(x) = x^2 - x\sin x - \cos x$

We have $f'(x) = 2x - x\cos x - \sin x + \sin x = x(2 - \cos x)$

The sign scheme for f' is given by

–	0	+
Decreasing		Increasing

Thus f vanishes for two values of x.

2nd solution : $f(x) = x^2 - x\sin x - \cos x$, $f'(x) = x(2 - \cos x)$

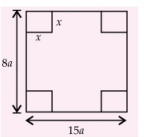
$x \to \infty,\ f(x) \to \infty$

$x \to -\infty,\ f(x) \to \infty$

Also, $f(0) = -1$

Hence $f(x)$ vanishes once in the interval $(-\infty, 0)$ and once in $(0, \infty)$.

19. (a, c) :



The volume is given by $V(x) = (15a - 2x)(8a - 2x)x$

And it is given that V_{max} is given at $x = 5$ *i.e.*, $2x = 10$

We have $V(t) = (15a - t)(8a - t)\dfrac{t}{2}$

$$= \frac{1}{2}\{t^2 - 23at + 120a^2\}t = \frac{1}{2}\{t^3 - 23t^2a + 120ta^2\}$$

$$\frac{dV}{dt} = \frac{1}{2}\{3t^2 - 46ta + 120a^2\}$$

Now, $\left.\dfrac{dV}{dt}\right|_{t=10} = 0 \Rightarrow 120a^2 - 46 \cdot 10 \cdot a + 3 \cdot 100 = 0$

$\Rightarrow\ 12a^2 - 46a + 30 = 0 \quad \Rightarrow\ 6a^2 - 23a + 15 = 0$

$\Rightarrow\ 6a^2 - 18a - 5a + 15 = 0 \quad \Rightarrow\ 6a(a - 3) - 5(a - 3) = 0$

$\Rightarrow\ (6a - 5)(a - 3) = 0. \quad \therefore\ a = 3, 5/6$

Also, $\dfrac{d^2V}{dt^2} = \dfrac{1}{2}\{6t - 46a\} = 3t - 23a$

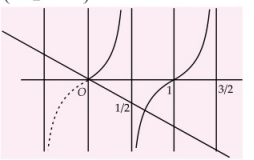
$\left.\dfrac{d^2V}{dt^2}\right|_{t=10,\ a=3} < 0$. Thus V is maximum.

Thus sides are 24 and 45.

20. (b, c) : $f(x) = x\sin\pi x \Rightarrow f'(x) = \sin\pi x + \pi x\cos\pi x$

$f'(x) = 0 \ \Rightarrow\ \tan\pi x = -\pi x \ \Rightarrow\ \pi x \in \left(\dfrac{2n+1}{2}\pi,\ n\pi + \pi\right)$

i.e., $x \in \left(n + \dfrac{1}{2},\ n + 1\right)$

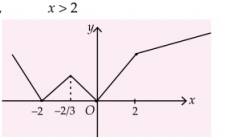


As $\left(n + \dfrac{1}{2},\ n + 1\right) \le (n, n + 1)$, we get two answers.

21. (a, d) : 1st solution : We have

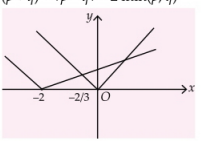
$f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$

$$= \begin{cases} -2x - 4, & x \le -2 \\ 2x + 4, & -2 < x \le -2/3 \\ -4x, & -\dfrac{2}{3} < x \le 0 \\ 4x, & 0 < x \le 2 \\ 2x + 4, & x > 2 \end{cases}$$



It has local minima at $x = -2$, 0 and maxima at –2/3.

2nd solution : We can draw the graph quite easily if we realise that $(p + q) - |p - q| = 2\min(p, q)$



So $f(x) = 2\min\{2|x|,\ |x + 2|\}$

22. (d): Observe that

$$e^{-x}\{f'' - 2f' + f\} = \frac{d}{dx}\{e^{-x}(f'(x) - f(x)\}$$

The given condition is

$$f''(x) - 2f'(x) + f(x) \ge e^x \Rightarrow e^{-x}\{f''(x) - 2f'(x) + f(x)\} \ge 1$$

$$\Rightarrow\ \frac{d}{dx}\{e^{-x}(f'(x) - f(x)\} \ge 0 \ \Rightarrow\ \frac{d^2}{dx^2}\{e^{-x}f(x)\} \ge 0$$

Let $g(x) = e^{-x}f(x)$

$g''(x) > 0$ means that g is concave upward.

$g(0) = g(1) = 0$

$\therefore\ g|x| < 0\ \forall\ x \in (0, 1) \Rightarrow e^{-x}f(x) < 0\ \forall\ x \in (0, 1)$

$\therefore \quad f(x) < 0$

23. (c) : We have $\frac{d}{dx}\{e^{-x}(f'(x) - f(x)\} \geq 0$

Let $A(x) = e^{-x} f(x)$

Then $A'(x) = e^{-x}(f'(x) - f(x))$

$A(x)$ has local minimum at $x = \frac{1}{4}$

$\therefore \quad A'(1/4) = 0$

Let $B(x) = e^{-x}\{f'(x) - f(x)\}$

B is an increasing function.

$B(x) < 0$ in $(0, 1/4)$

$\Rightarrow \quad e^{-x}(f'(x) - f(x)) < 0$ in $(0, 1/4)$

Thus $f'(x) < f(x)$ in $(0, 1/4)$

24. (9) : Let $p'(x) = \lambda(x-1)(x-3)$

$$p(x) = \lambda\left(\frac{x^3}{3} - 2x^2 + 3x\right) + k$$

$p(1) = 6 \quad \Rightarrow \quad \frac{4}{3}\lambda + k = 6$

$p(3) = 2 \quad \Rightarrow \quad k = 2$

Thus $\lambda = 3$

$p'(0) = 3\lambda = 9$

25. (5) : $f(x) = |x| + |x^2 - 1|$

We have $f(x) = \begin{cases} -x + x^2 - 1, & x \leq -1 \\ -x + 1 - x^2, & -1 < x \leq 0 \\ x + 1 - x^2, & 0 < x \leq 1 \\ x + x^2 - 1, & x > 1 \end{cases}$

Then $f(x) = \begin{cases} x^2 - x - 1, & x \leq -1 \\ -x^2 - x + 1, & -1 < x \leq 0 \\ -x^2 + x + 1, & 0 < x \leq 1 \\ x^2 + x - 1, & x > 1 \end{cases}$

$$f'(x) = \begin{cases} 2x - 1, & x < -1 \\ -2x^2 - 1, & -1 < x < 0 \\ -2x^2 + 1, & 0 < x < 1 \\ 2x + 1, & x > 1 \end{cases}$$

$f'(x) = 0 \Rightarrow x = \frac{1}{2}, -\frac{1}{2}$

As f' doesn't exist at $x = 1$ or $-1, 0$

Hence, there are 5 such points.

26. (2) : $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$\Rightarrow \quad f'(x) = 4x^3 - 12x^2 + 24x + 1$

$\Rightarrow \quad f''(x) = 12x^2 - 24x + 24 = 12(x^2 - 2x + 2)$

$= 12\{(x-1)^2 + 1\} > 0 \ \forall \ x \in R$

Thus $f'(x)$ is a strictly increasing function. Let x_0 be a real root of $f'(x) = 0$

Thus $f(x)$ is monotonically decreasing for $x \in (-\infty, x_0)$ and monotonically increasing for (x_0, ∞) where $x_0 < 0$.

$f(0) = -1 < 0, f(1) = 1 - 4 + 12 + 1 - 1 = 9 > 0$

$f(-1) = 1 + 4 + 12 - 1 - 1 = 15 > 0$

Thus $f(-1) > 0, f(0) < 0, f(1) > 0$

Hence it has two real and distinct roots.

27. (d): Rewrite the given equation as

$f'(x) < 2f(x) \quad \Rightarrow \quad f'(x) - 2f(x) < 0$

$\Rightarrow \quad e^{-2x}(f'(x) - 2f(x)) < 0 \quad \Rightarrow \quad \frac{d}{dx}(e^{-2x} f(x)) < 0$

Put $g(x) = e^{-2x} f(x)$. The condition thus reach $g'(x) < 0$.

Thus g is decreasing on $\left[\frac{1}{2}, 1\right]$.

Hence, $x > \frac{1}{2} \quad \Rightarrow \quad g(x) < g\left(\frac{1}{2}\right)$

$\Rightarrow \quad e^{-2x} f(x) < e^{-1} f\left(\frac{1}{2}\right) \quad \Rightarrow \quad f(x) < e^{2x-1}$

On integrating both sides,

$$\int_{1/2}^{1} f'(x) < \int_{1/2}^{1} e^{2x-1}\,dx = \frac{e^{2x-1}}{2}\Big|_{1/2}^{1} = \frac{1}{2}(e-1)$$

As f is positive, we have $\int_{1/2}^{1} f(x)dx > 0$

Hence, the interval in which the integral lies is $\left(0, \frac{e-1}{2}\right)$.

28. (c) : $\sec x + \tan x = t$. Also $\sec x - \tan x = \frac{1}{t}$

Then $(\sec x \tan x + \sec^2 x)dx = dt$

So that $dx = \frac{dt}{\sec x \cdot t}$

The integrand $\int \frac{\sec^2 x\,dx}{(\sec x + \tan x)^{9/2}} = \int \frac{\left(t + \frac{1}{t}\right)}{2 \cdot t \cdot t^{9/2}}\,dt$

$= \frac{1}{2}\int t^{9/2} dt + \frac{1}{2}\int t^{-13/2} dt$

$= -\frac{1}{2}\left[\frac{2}{7t^{7/2}} + \frac{2}{11t^{11/2}}\right] = -\frac{1}{t^{11/2}}\left[\frac{t^2}{7} + \frac{1}{11}\right] + K$

where K is an arbitrary constant.

29. (b) : 1st solution : $I = \int_{-\pi/2}^{\pi/2} \left(x^2 + \ln\frac{\pi + x}{\pi - x}\right)\cos x\,dx$

As $g(x) = \ln\left(\frac{\pi + x}{\pi - x}\right)$ satisfies $g(x) = -g(-x)$. Thus, $g(x)$ is odd.

Hence $\int_{-\pi/2}^{\pi/2} \underbrace{g(x)\cos x}_{\text{odd}}\,dx = 0$

Now the integral reduces to $I = \int_{-\pi/2}^{\pi/2} x^2 \cos x dx$

(Integration by parts)

$$= 2\left\{(x^2 \sin x)_0^{\pi/2} - \int_0^{\pi/2} 2x \sin x dx\right\}$$

$$= 2\left(\frac{\pi^2}{4}\right) - 4\int_0^{\pi/2} x \sin x dx$$

$$= \frac{\pi^2}{4} - 4\left[(-x)(\cos x) + \int_0^{\pi/2} x \cos x dx\right]$$

$$= \frac{\pi^2}{2} - 4(-x\cos x + \sin x)_0^{\pi/2} = \frac{\pi^2}{4} - 4(1) = \frac{\pi^2}{4} - 4$$

2nd solution :

Using generalized formula for integration by parts

$$2\int_0^{\pi/2} x^2 \cos x dx = 2x[(x^2)(\sin x) - (2x)(-\cos x) + (2)(-\sin x)]_0^{\pi/2}$$

$$= 2\left[\frac{\pi^2}{4} - 2\right] = \frac{\pi^2}{2} - 4$$

30. (b): $f(x) = (1-x)^2 \sin^2 x + x^2,\ x \in R$

$$g'(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ln t\right) f(t) dt \ \forall\ x \in (1, \infty)$$

We have $g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) f(x) = A(x) f(x)$

Let $A(x) = \frac{2(x-1)}{x+1} - \ln x$

$$A'(x) = \frac{4}{(x+1)^2} - \frac{1}{x} = \frac{-(x-1)^2}{(x+1)^2 x} < 0$$

As $A(1) = 0 \ \therefore\ A(x) < 0\ \forall\ x > 1$

As $g'(x) = A(x) f(x) < 0$ $[\because A(x) < 0, f(x) < 0]$

Thus g is decreasing.

31. (c) : $f(x) + 2x = 2(1 + x^2)$

$\Rightarrow\ (1-x)^2 \sin^2 x + x^2 + 2x = 2(1 + x^2)$

$\Rightarrow\ (1-x)^2 \sin^2 x = x^2 - 2x + 2 = (1-x^2) + 1$

$\Rightarrow\ (1-x)^2 \cos^2 x + 1 = 0$

Not possible. Hence P is false.

Again let $2(1-x)^2 \sin^2 x + 2x^2 + 1 = 2x + 2x^2$

$\Rightarrow\ 2(1-x)^2 \sin^2 x = 2x - 1$

Thus $2\sin^2 x = \frac{2x-1}{(x-1)^2}$

Define $B(x) = \frac{2x-1}{(x-1)^2} - 2\sin^2 x$

$B(0) < 0,\ \lim_{x \to 1^-} B(1) \to -\infty$

Thus by intermediate value property (IVP) of continuous functions, the equation has a solution.

2nd solution : Define $M(x) = 2f(x) + 1 - 2x(1 + x)$

We have $M(0) = 2f(0) + 1 - 0 = 1 > 0$

$M(1) = 2f(1) + 1 - 4 = -3 < 0$

Thus by IVP of continuous function $M(x) = 0$ has a solution. So statement Q is true.

32. (a, b, c, d) : $f(x) = \int_0^x e^{t^2} (t-2)(t-3) dt$

We have $f'(x) = e^{x^2} (x-2)(x-3)$

The sign scheme for f' is

The f has a local maxima at $x = 2$ and a local minima at $x = 3$.

Also f is decreasing on (2, 3)

As $f'(2) = 0$ and $f'(3) = 0$

Using Rolle's theorem, $\exists\ c \in (2, 3)$ such that $f''(c) = 0$.

33. (a) : Set $x^2 = u$ to obtain

$$I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$$

$$= \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin u}{\sin u + \sin(\ln 6 - u)} du \qquad \text{.....(i)}$$

Using $\int_a^b f(x)\, dx = \int_a^b f(a+b-x) dx$, we have

$$I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - u)}{\sin u + \sin(\ln 6 - u)} du \qquad \text{.....(ii)}$$

Adding (i) and (ii), we get

$$2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} du = \frac{1}{2} \cdot \ln\frac{3}{2} \Rightarrow\ I = \frac{1}{2}\ln\frac{3}{2}.$$

34. (b): The curves are

$y = \sin x + \cos x,\ x \in [0, \pi/2]$

$$y = \begin{cases} \cos x - \sin x, & x \in [0, \pi/4) \\ \sin x - \cos x, & x \in (\pi/4, \pi/2] \end{cases}$$

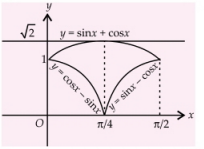
The area required is set up by the integral

$$\int_0^{\pi/4} \{(\sin x + \cos x) - (\cos x - \sin x)\}\, dx + \int_{\pi/4}^{\pi/2} \{(\sin x + \cos x) - (\sin x - \cos x)\} dx$$

$$= 2\int_0^{\pi/4} \sin x dx + 2\int_{\pi/4}^{\pi/2} \cos x dx = 2(-\cos x)_0^{\pi/4} + 2(\sin x)_{\pi/4}^{\pi/2}$$

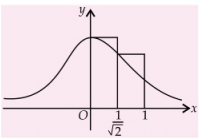
$$= 2\left(1 - \frac{1}{\sqrt{2}}\right) + 2\left(1 - \frac{1}{\sqrt{2}}\right) = 4\left(1 - \frac{1}{\sqrt{2}}\right) = 2\sqrt{2}\,(\sqrt{2} - 1)$$

A rough sketch is as under



35. (a, b, d) : $S = \int_0^1 e^{-x^2}\,dx$

As $0 \le x^2 \le 1$



$\Rightarrow \quad -1 \le -x^2 \le 0$

$\frac{1}{e} \le e^{-x^2} \le 1$

$S \ge \int_0^1 \frac{1}{e}\,dx \Rightarrow S \ge \frac{1}{e}$

As $e^{-x^2} \ge e^{-x},\ \forall\, x \in (0,1)$

We have $S \ge \int_0^1 e^{-x}dx = [-e^{-x}]_0^1 = 1 - \frac{1}{e}$

Again area of two rectangles

$= 1\left(\frac{1}{\sqrt{2}} - 0\right) + e^{-\frac{1}{2}}\left(1 - \frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

Thus $S \le \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{e}}\left(1 - \frac{1}{\sqrt{2}}\right)$

36. (b) : $R_1 = \int_0^b (1-x)^2\,dx = -\frac{(1-x)^3}{3}\Big]_0^b = \frac{(b-1)^3 + 1}{3}$

$R_2 = \int_b^1 (x-1)^2\,dx = \frac{(x-1)^3}{3}\Big]_b^1 = -\frac{(b-1)^3}{3}$

We have $R_1 - R_2 = \frac{1}{4}$

$\Rightarrow \quad \frac{(b-1)^3 + 1}{3} + \frac{(b-1)^3}{3} = \frac{1}{4} \Rightarrow \frac{2(b-1)^3}{3} = \frac{1}{4} - \frac{1}{3}$

$\Rightarrow \quad \frac{2(b-1)^3}{3} = \frac{3-4}{12} = -\frac{1}{12} \Rightarrow (b-1)^3 = -\frac{1}{8}$

$\Rightarrow \quad (b-1) = -\frac{1}{2} \ \therefore b = \frac{1}{2}$

37. (c) : $R_2 \ge \int_{-1}^{2} f(x)\,dx$

$R_1 = \int_{-1}^{2} xf(x)dx = \int_{-1}^{2} (1-x)f(1-x)dx$

We have on addition

$2R_1 = \int_{-1}^{2} f(x)dx$ as $f(x) = f(1-x) = R_2$.

38. (a) : 1st solution : Given $\frac{dy}{dx} = \frac{y}{x} + \sec\frac{y}{x}$

Let $y = vx$, as it is homogeneous.

$$v + x\frac{dv}{dx} = v + \sec v \quad \Rightarrow \quad \cos v\,dv = \frac{dx}{x}$$

$\Rightarrow \quad \sin v = \ln x + \lambda$ (On integration)

$\ln x + \lambda = \sin\frac{y}{x}$

As the curve passes through $(1, \pi/6)$ we have $\lambda = 1/2$

Hence $\sin\frac{y}{x} = \ln x + \frac{1}{2}$

2nd solution : Using inspection

$\frac{x\,dy - y\,dx}{x} = \sec\frac{y}{x}dx \quad \Rightarrow \quad \cos\frac{y}{x}\frac{x\,dy - y\,dx}{x^2} = \frac{dx}{x}$

$\Rightarrow \quad \cos\frac{y}{x}\,d(y/x) = \frac{dx}{x}$

On integration, $\sin\frac{y}{x} = \ln x + \lambda$, etc.

39. (a, d) : $\frac{dy}{dx} - y\tan x = 2x\sec x$

It is a linear differential equation.

$\therefore \quad$ I.F. $= e^{-\int \tan x dx} = e^{-\ln(\sec x)} = \cos x$

The solution reads

$y \cdot \cos x = \int 2x\sec x\cos x\,dx = x^2 + c$

We have $y(0) = 0 \ \Rightarrow \ c = 0 \ \therefore \ y = x^2\sec x$

$y\left(\frac{\pi}{4}\right) = \frac{\pi^2}{16}\cdot\sqrt{2} = \frac{\pi^2}{8\sqrt{2}}$

$y' = 2x\sec x + x^2\sec x\tan x$

$y'\left(\frac{\pi}{3}\right) = 2\cdot\frac{\pi}{3}\cdot 2 + \frac{\pi^2}{9}2\cdot\sqrt{3} = \frac{4\pi}{3} + \frac{2\pi^2\sqrt{3}}{9}$

20. (6) : THE QUESTION IS WRONG AS $f(1) = 2$ contradicts to the hypothesis. $f(1)$ turns out to be $f(1) = 1/3$ by setting $v = 1$ in the problem.
However solution goes like this.

1st solution :

$6\int_1^x f(t)dt = 3xf(x) - x^3$. We have on differentiating

$6f(x) = 3xf'(x) + 3f(x) - 3x^2$

$\Rightarrow \quad f(x) = xf'(x) - x^2 \Rightarrow \ y = x\frac{dy}{dx} - x^2 \ \Rightarrow \ \frac{dy}{dx} - \frac{1}{x}y = x$

I.F. $= e^{\int -\frac{1}{x}dx} = e^{-\ln x} = \frac{1}{x}$

General solution is $y.\frac{1}{x}=\int x.\frac{1}{x}dx+k=x+k$

$y=x^2+kx$, $x=1$ and $y=2 \Rightarrow k=1$

Thus $y(x)=x^2+x$, $y(2)=2^2+2=6$

2nd solution : We have $y=x\frac{dy}{dx}-x^2$

Now make use of inspection method to obtain that

$xdy-ydx=x^2dx \Rightarrow \frac{xdy-ydx}{x^2}=dx \Rightarrow d\left(\frac{y}{x}\right)=dx$

Thus $\frac{y}{x}=x+\lambda$

$\Rightarrow \quad y=x^2+\lambda x$, as before in solution 1.

40. (0) : Given $y'(x)+y(x)g'(x)=g(x)g'(x)$, $y(0)=0$

Integrating factor $=e^{\int g'(x)dx}=e^{g(x)}$

Solution is $y(x)e^{g(x)}=\int e^{g(x)}g(x)g'(x)dx+k$

Let $g(x)=v$, then $y(x)e^{g(x)}=\int e^v v dv+k=ve^v-e^v+k$

Thus $g(x)=(g(x)-1)+ke^{-g(x)}$

$x=0$, $y(0)=(g(0)-1)+ke^{-g(0)}$ giving $k=1$

$y(2)=(g(2)-1)+e^{-g(2)}$

Thus $y(2)=0$

41. (c) : The sides of $\overline{PS}$ and $\overline{PQ}$ are given by

$$\overline{PS}=\frac{\overline{PR}-\overline{SQ}}{2} \text{ and } \overline{PQ}=\frac{\overline{PR}+\overline{SQ}}{2}$$

The volume V is given by

$$V=\left|\left[\overline{PQ}\ \overline{PS}\ \overline{PT}\right]\right|=\frac{1}{4}\left|\left[\overline{PR}+\overline{SQ},\ \overline{PR}-\overline{SQ},\ \overline{PT}\right]\right|$$

$$=\frac{1}{2}\left|\left[\overline{PR},\ \overline{SQ},\ \overline{PT}\right]\right|$$

[Recall that $(\vec{a}\ \vec{b}\ \vec{b})=0$ and $(\vec{a}\ \vec{b}\ \vec{c})=[\vec{b}\ \vec{c}\ \vec{a}]$, etc.]

$$=\frac{1}{2}\begin{vmatrix}3 & 1 & -2\\ 1 & -3 & -4\\ 1 & 2 & 3\end{vmatrix}=\frac{1}{2}|[3(-1)-1(7)-2(5)|$$

$$=\frac{1}{2}|(-20)|=10$$

Remark : One could have also calculated the sides $\overline{PS}$ and $\overline{PQ}$ and then proceeded to evaluate the box product using determinants.

42. (5) : 1st solution : Let vectors be given as

$\vec{a}=(1,1,1)$ $\qquad$ $\vec{a}'=(-1,-1,-1)$

$\vec{b}=(-1,1,1)$ $\qquad$ $\vec{b}'=(1,-1,-1)$

$\vec{c}=(1,-1,1)$ $\qquad$ $\vec{c}'=(-1,1,-1)$

$\vec{d}=(1,1,-1)$ $\qquad$ $\vec{d}'=(-1,-1,1)$

$\vec{a}$ and $\vec{a}'$ with any of the remaining six will produce a set of three coplanar vectors.

Number of such choices $=4\times 6=24$

The number of non coplanar vectors $={}^8C_3-24$

$=56-24=32=2^5$

2nd solution : The four vectors of V form four diagonals of a cube and remaining four are (anti) parallel.

Number of ways of selecting three non-coplanar vectors $={}^4C_3\times 2\times 2\times 2=2^5$

3rd solution : The required number is given by

$$\frac{8\cdot 6\cdot 4}{3!}=32$$

43. (a, d) : $\frac{x-5}{0}=\frac{y-0}{3-\alpha}=\frac{z-0}{2}$

and $\frac{x-\alpha}{0}=\frac{y-0}{-1}=\frac{z-0}{2-\alpha}$

are the given lines. Then lines will be coplanar if

$$\begin{vmatrix}5-\alpha & 0 & 0\\ 0 & 3-\alpha & -2\\ 0 & -1 & 2-\alpha\end{vmatrix}=0$$

$\Rightarrow \quad (5-\alpha)\{(3-\alpha)(2-\alpha)-2\}=0$

$\Rightarrow \quad (5-\alpha)(\alpha^2-5\alpha+4)=0 \quad \therefore \quad \alpha=1, 4, 5$

44. (c) : (P) $[\vec{a}\ \vec{b}\ \vec{c}]=2$

$[2(\vec{a}\times\vec{b}), 3(\vec{b}\times\vec{c}), (\vec{c}\times\vec{a})]=6[\vec{a}\ \vec{b}\ \vec{c}]^2=6\times 4=24$

(Q) $[\vec{a}\ \vec{b}\ \vec{c}]=5$

$[3(\vec{a}+\vec{b}), (\vec{b}+\vec{c}), 2(\vec{c}+\vec{a})]=6\times 2[\vec{a}\ \vec{b}\ \vec{c}]$

$=6\times 2\times 5=60$

(R) $\frac{1}{2}|\vec{a}\times\vec{b}|=20=\frac{1}{2}|(2\vec{a}+3\vec{b})\times(\vec{a}-\vec{b})|$

$=\frac{5}{2}(\vec{a}\times\vec{b})=\frac{5}{2}\times 40=100$

(S) $|\vec{a}\times\vec{b}|=30$

$\Rightarrow \quad |(\vec{a}+\vec{b})\times\vec{a}|=|\vec{b}\times\vec{a}|=30$

45. (3) : 1st solution :

We have $|\vec{a}-\vec{b}|^2+|\vec{b}-\vec{c}|^2+|\vec{c}-\vec{a}|^2$

$=3\{|a|^2+|b|^2+|c|^2\}-(\vec{a}+\vec{b}+\vec{c})^2=9$

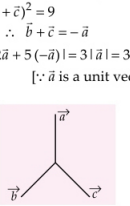
$\Rightarrow \quad 3\times 3=(\vec{a}+\vec{b}+\vec{c})^2=9$

$\Rightarrow \quad \vec{a}+\vec{b}+\vec{c}=0 \quad \therefore \quad \vec{b}+\vec{c}=-\vec{a}$

$|2\vec{a}+5\vec{b}+5\vec{c}|=|2\vec{a}+5(-\vec{a})|=3|\vec{a}|=3$

[$\because \vec{a}$ is a unit vector]

2nd solution :



Angle between each pair is 120°. Then $\vec{a}+\vec{b}+\vec{c}=0$

Now $|2\vec{a}+5\vec{b}+5\vec{c}| = |2\vec{a}-5\vec{a}| = 3|\vec{a}| = 3$

46. (c) : $\vec{a}\times(2\hat{i}+3\hat{j}+4\hat{k}) = (2\hat{i}+3\hat{j}+4\hat{k})\times\vec{b}$

$\Rightarrow \quad (\vec{a}+\vec{b})\times(2\hat{i}+3\hat{j}+4\hat{k}) = 0$

$\therefore \quad \vec{a}+\vec{b}$ is parallel to $2\hat{i}+3\hat{j}+4\hat{k}$

$\therefore \quad \vec{a}+\vec{b} = \lambda(2\hat{i}+3\hat{j}+4\hat{k})$

$(\vec{a}+\vec{b})\times(-7\hat{i}+2\hat{j}+3\hat{k}) = \lambda(2\hat{i}+3\hat{j}+4\hat{k})(-7\hat{i}+2\hat{j}+3\hat{k})$

$= \lambda(-14+6+12) = 4\lambda$

But $\lambda = 1, -1$

Hence a possible value of the desired thing is 4.

47. (a, d) : 1st solution : Observe that $\vec{c}\times(\vec{a}\times\vec{b})$ is a vector in the plane of $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$.

Let $\vec{a} = \hat{i}+\hat{j}+2\hat{k}$, $\vec{b} = \hat{i}+2\hat{j}+\hat{k}$, $\vec{c} = \hat{i}+\hat{j}+\hat{k}$.

So a vector satisfying our requirements is

$\vec{c}\times(\vec{a}\times\vec{b}) = (\vec{c}\cdot\vec{b})\vec{a} - (\vec{c}\cdot\vec{a})\vec{b}$

$= (1+2+1)(\hat{i}+\hat{j}+2\hat{k}) - (1+1+2)(\hat{i}+2\hat{j}+\hat{k})$

$= 4(\hat{i}+\hat{j}+2\hat{k}) - 4(\hat{i}+2\hat{j}+\hat{k}) = -4\hat{j}+4\hat{k} = 4(-\hat{j}+\hat{k})$

Any scalar multiple of the above vector is a solution to our problem.

2nd solution :

Any vector in the plane of $\hat{i}+\hat{j}+2\hat{k}$ are $\hat{i}+2\hat{j}+\hat{k}$ is

$\vec{\gamma} = \lambda(\hat{i}+\hat{j}+2\hat{k}) + \mu(\hat{i}+2\hat{j}+\hat{k})$

$= (\lambda+\mu)\hat{i} + (\lambda+2\mu)\hat{j} + (2\lambda+\mu)\hat{k}$

Again $\vec{\gamma}\cdot\vec{c} = 0$ with $\vec{c} = \hat{i}+\hat{j}+\hat{k}$ gives $\lambda+\mu = 0$

So $\vec{\gamma} = -\lambda\hat{j}+\lambda\hat{k} = \lambda(-\hat{j}+\hat{k})$

48. (d): Let $\dfrac{x+2}{2} = \dfrac{y+1}{-1} = \dfrac{z}{3} = k$

then $(2k-2, -k-1, 3k)$ is a point on the line.

Let the foot of the perpendicular from this point on the plane $x+y+z-3=0$ be (α, β, γ) then it is given by

$$\frac{\alpha-(2k-2)}{1} = \frac{\beta-(-k-1)}{1} = \frac{\gamma-3k}{1}$$

$$= -\frac{(2k-2)+(-k-1)+(3k)-3}{3} = \frac{6-4k}{3}$$

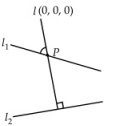
Thus, $\alpha = \dfrac{2k}{3}$, $\beta = 1-\dfrac{7k}{3}$, $\gamma = 2+\dfrac{5k}{3}$ then the line is given by

$$\frac{x}{2/3} = \frac{y-1}{-7/3} = \frac{z-2}{5/3} \quad i.e., \ \frac{x}{2} = \frac{y-1}{-7} = \frac{z-2}{5}$$

Remark : One can also take two points say $(-4, 0, -3)$ and $(0, -2, 3)$ and then find the foot of perpendicular of these two points on the given plane. The line passing through these two points will give the locus required.

49. (b, d): The line l is perpendicular to l_1 and l_2. Hence the direction ratios of l are given by the vector

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 2 \\ 2 & 2 & 1 \end{vmatrix} = -2\hat{i}+3\hat{j}-2\hat{k}$$



Let $P \equiv (2r, -3r, 2r)$

As it lies in l_1, we have

$$\frac{2r-3}{1} = \frac{-3r+1}{2} = \frac{2r-4}{2}$$

Thus we have $r = 1$

And $P \equiv (2, -3, 2)$

A point on l_2 is $(3+2s, 3+2s, 2+s)$

Then $(3+2s-2)^2 + (3+2s+3)^2 + (2+s-2)^2 = 17$

$\Rightarrow \quad 9s^2+28s+20 = 0$ giving $s = -2, -10/9$

Thus the points are $(-1, -1, 0)$ and $\left(\dfrac{7}{9}, \dfrac{7}{9}, \dfrac{8}{9}\right)$.

50. (a) : The normal to the plane is

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 7 & 1 & 2 \\ 3 & 5 & -6 \end{vmatrix} = -16\hat{i}+48\hat{j}+32\hat{k}$$

$\therefore \quad \vec{h} = \hat{i}-3\hat{j}-2\hat{k}$

The point of intersection of L_1 and L_2 is given by

$2\lambda+1 = \mu+4$

$-\lambda = \mu-3$

$1 = 3\mu-2$

So, $\mu = 1$

$\therefore$ Point of intersection is $(5, -2, -1)$

The equation of desired plane is

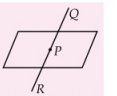
$(x-5)-3(y+2)-2(z+1) = 0$

$\Rightarrow \quad x-3y-2z = 13$

51. (a) : 1st solution :

Let equation of line QR be

$$\frac{x-2}{1} = \frac{y-3}{4} = \frac{z-5}{1}$$



Let the point P be given parametrically as

$P(2+t, 3+4t, 5+t)$

As P lies in the plane $5x-4y-z = 1$

We have $5(2+t)-4(3+4t)-(5+t) = 1$

$\Rightarrow \quad 10+5t-12-16t-5-t = 1$

$\Rightarrow \quad -12t-7 = 1$

$\Rightarrow \quad 12t = -8 \quad \therefore \quad t = -\dfrac{8}{12} = -\dfrac{2}{3}$

The co-ordinates of P are $\left(\dfrac{4}{3}, \dfrac{1}{3}, \dfrac{13}{3}\right)$

Again let $S \equiv (2+m, 3+4m, 5+m)$ for being a parameter

As S is the foot of perpendicular, we have

$\therefore \quad 1(2+m-2)+4(3+4m-1)+1(5+m-4)=0$

$\Rightarrow \quad m+4(4m+2)+m+1=0$

$\Rightarrow \quad 18m+9=0 \ \therefore \quad m=-\frac{1}{2}$

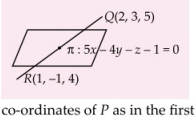
Thus $S \equiv \left(\frac{3}{2}, 1, \frac{9}{2}\right)$

We have by distance formula

$$PS=\sqrt{\left(\frac{3}{2}-\frac{4}{3}\right)^2+\left(1-\frac{1}{3}\right)^2+\left(\frac{9}{2}-\frac{13}{3}\right)^2}$$

$$=\sqrt{\left(\frac{1}{6}\right)^2+\left(\frac{2}{3}\right)^2+\left(\frac{1}{6}\right)^2}=\sqrt{\frac{1}{36}+\frac{4}{9}+\frac{1}{36}}=\sqrt{\frac{9}{18}}=\frac{1}{\sqrt{2}}$$

2nd solution :



We find the co-ordinates of P as in the first solution

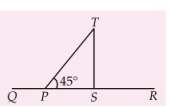
$P \equiv \left(\frac{4}{3}, \frac{1}{3}, \frac{13}{3}\right)$

The d.r'.s of TP are (2, 2, –1)

Angle between RQ and PT is given by

$$\cos\theta=\frac{(1)(2)+(4)(2)+(1)(-1)}{\sqrt{1^2+4^2+1^2}\sqrt{2^2+2^2+(-1)^2}}$$

$$=\frac{2+8-1}{\sqrt{18}\sqrt{9}}=\frac{9}{3\sqrt{2}\cdot 3}=\frac{1}{\sqrt{2}}$$



Thus $\theta = 45°$

As $PT = 1$, we have

$PS = TS = \frac{1}{\sqrt{2}}$

52. (a) : The plane through the intersection of $x+2y+3z-2=0$ and $x-y+z-3=0$ is

$(x+2y+3z-2)+k(x-y+z-3)=0$

$\Rightarrow \quad (1+k)x+(2-k)y+(3+k)z-(2+3k)=0$

The distance of the above plane from (3, 1, –1) is

$$\frac{2}{\sqrt{3}}=\frac{|3(1+k)+(2-k)-(3+k)-(2+3k)|}{\sqrt{(k+1)^2+(2-k)^2+(3+k)^2}}$$

$$\Rightarrow \quad \frac{4}{3}=\frac{4k^2}{3k^2+4k+14}$$

$\Rightarrow \quad 12k^2+16k+56=12k^2 \Rightarrow 8(2k+7)=0$

$\therefore \quad k=-7/2$

Thus the desired plane is $-\frac{5}{2}x+\frac{11}{2}y-\frac{z}{2}+\frac{17}{2}=0$

$\therefore \quad 5x-11y+z=17$

53. (b, c) : For the straight lines to be coplanar the determinant $\begin{vmatrix} 2 & k & 2 \\ 5 & k & 2 \\ 2 & 0 & 0 \end{vmatrix} = 0$

$\Rightarrow \quad k^2-4=0 \quad \therefore k=2, -2$

For $k = 2$, we have lines as $\frac{x-1}{2}=\frac{y+1}{2}=\frac{z}{2}$

and $\frac{x+1}{5}=\frac{y+1}{2}=\frac{z}{2}$

Easily we see that $y + 1 = z$ is the plane that contains both.

For $k = –2$, let the plane containing the first line be

$(x-z-1)+\mu(x+y)=0$

The point(–1, –1, 0) lies on it, then

$(-1-1)+\mu(-1-1)=0$

$\Rightarrow \quad -2-2\mu=0 \qquad \therefore \mu=-1$

Thus the equation is $x-z-1-x-y=0$

$\therefore \quad y+z+1=0$

54. (c) : Let $\vec{v}=l\vec{a}+m\vec{b}$ for some scalars l and m.

$\vec{v}=(l+m)\hat{i}+(l-m)\hat{j}+(l+m)\hat{k}$

Also $\vec{v}\cdot\hat{c}=\frac{1}{\sqrt{3}}$ gives $\frac{(l+m)-(l-m)-(l+m)}{\sqrt{3}}=\frac{1}{\sqrt{3}}$

$\Rightarrow \quad m-l=1$. For $l = 1$, $m = 2$

And then $\vec{v}$ becomes $\vec{v}=3\hat{i}-\hat{j}+3\hat{k}$

55. (9) : Given $\vec{r}\times\vec{b}=\vec{c}\times\vec{b}=(\vec{r}-\vec{c})\times\vec{b}=0$

Thus $\vec{r}-\vec{c}$ is parallel to $\vec{b}$

$\vec{r}-\vec{c}=t\vec{b} \ \Rightarrow \ \vec{r}=\vec{c}+t\vec{b}$, t a scalar

$(\vec{c}+t\vec{b})\cdot\vec{a}=0$

$\Rightarrow \quad ((\hat{i}+2\hat{j}+3\hat{k})+t(-\hat{i}+\hat{j}))\cdot(-\hat{i}-\hat{k})=0$

$\Rightarrow \quad t-4=0 \ \therefore \ t=4.$

Now $\vec{r}\cdot\vec{b}=(-3\hat{i}+6\hat{j}+3\hat{k}).(-\hat{i}+\hat{j})=3+6=9$

56. (a) : P(The problem is solved by at least one person)

$= 1 - P$(the problem is solved by none of the persons)

$$=1-\left(1-\frac{1}{2}\right)\left(1-\frac{3}{4}\right)\left(1-\frac{1}{4}\right)\left(1-\frac{1}{8}\right)$$

$$=1-\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{7}{8}=1-\frac{21}{256}=\frac{235}{256}$$

57. (6) : 1st solution : Let x, y and z be the probabilities of E_1, E_2 and E_3. Then

$x(1-y)(1-z)=\alpha$; $\quad y(1-x)(1-z)=\beta$;

$z(1-x)(1-y)=\gamma$

Also, $(1-x)(1-y)(1-z)=p$

On dividing, we get, $\frac{x}{1-x}=\frac{\alpha}{p}$. $\quad \therefore \quad x=\frac{\alpha}{\alpha+p}$ etc.

$$\frac{P(E_1)}{P(E_3)}=\frac{\frac{\alpha}{\alpha+p}}{\frac{\gamma}{\gamma+p}}=\frac{1+\frac{p}{\gamma}}{1+\frac{p}{\alpha}}$$

Again, $(\alpha-2\beta)p=\alpha\beta$; $(\beta-3\gamma)p=2\beta\gamma$

We have $\alpha p=\beta(\alpha+2p)$ and $3\gamma p=\beta(p-2\gamma)$

From above, $\frac{\alpha}{3\gamma}=\frac{\alpha+2p}{p-2\gamma}$

$$\Rightarrow \quad \frac{\alpha+2p}{\alpha}=\frac{p-2\gamma}{3\gamma} \quad \Rightarrow \quad 1+\frac{2p}{\alpha}=\frac{p}{3\gamma}-\frac{2}{3}$$

$$\Rightarrow \quad \frac{5}{3}=\frac{p}{3\gamma}-\frac{2p}{\alpha} \quad \Rightarrow \quad 5=\frac{p}{\gamma}-\frac{6p}{\alpha}$$

$$\Rightarrow \quad 6\left(1+\frac{p}{\alpha}\right)=1+\frac{p}{\gamma}. \qquad \therefore \quad \frac{1+\frac{p}{\gamma}}{1+\frac{p}{\alpha}}=6$$

2nd solution : The algebra can be handled differently. Use $(\alpha-2\beta)p=\alpha\beta$ we have

$\{x(1-y)(1-z)-2y(1-x)(1-z)\}\ \{(1-x)(1-y)(1-z)\}$

$= xy(1-x)(1-y)(1-z)^2$

which gives $x=2y$. Similarly, $y=3z$

Now, $x=6z$. $\quad \therefore \quad \frac{x}{z}=6$

58. (a) : Probability that the three drawn balls are of same colour = P(WWW) + P(RRR) + P(BBB)

$$=\frac{1}{6}\cdot\frac{2}{9}\cdot\frac{3}{12}+\frac{3}{6}\cdot\frac{3}{9}\cdot\frac{4}{12}+\frac{2}{6}\cdot\frac{4}{9}\cdot\frac{5}{12}$$

$$=\frac{6+36+40}{6\cdot 9\cdot 12}=\frac{82}{648}$$

59. (d): The required probability

$$=\frac{\frac{1}{3}\cdot\frac{2\cdot 3}{{}^{9}C_2}}{\frac{1}{3}\left\{\frac{1\cdot 3}{{}^{6}C_2}+\frac{2\cdot 3}{{}^{9}C_2}+\frac{3\cdot 4}{{}^{12}C_2}\right\}}=\frac{\frac{1}{6}}{\frac{1}{5}+\frac{1}{6}+\frac{2}{11}}$$

$$=\frac{\frac{1}{6}}{\frac{55+66+60}{55\cdot 6}}=\frac{55}{181}$$

60. (b, d): Given $P(X_1)=\frac{1}{2}$,

$P(X_2)=\frac{1}{4}$ and $P(X_3)=\frac{1}{4}$

$P(X) = P(E_1E_2E_3) + P(\bar{E}_1E_2E_3) + P(E_1\bar{E}_2E_3) + P(E_1E_2\bar{E}_3)$

$$=\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}$$

$$=\frac{1}{32}+\frac{1}{32}+\frac{3}{32}+\frac{3}{32}=\frac{8}{32}=\frac{1}{4}$$

P(exact two functioning $|\ X$)

$$=\frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}}{\frac{1}{4}}$$

$$=\frac{\frac{1}{32}+\frac{3}{32}+\frac{3}{32}}{\frac{1}{4}}=\frac{7}{32}\cdot\frac{4}{1}=\frac{7}{8}$$

$$P(X\mid X_1)=\frac{\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{3}{4}\cdot\frac{1}{4}+\frac{1}{2}\cdot\frac{1}{4}\cdot\frac{3}{4}}{\frac{1}{2}}=\frac{\frac{1+3+3}{32}}{\frac{1}{2}}$$

$$=\frac{7}{32}\cdot 2=\frac{7}{16}$$

61. (a, b) : $P(X\mid Y)=\frac{1}{2}$

$$\Rightarrow \quad \frac{P(X\cap Y)}{P(X)}=\frac{1}{2} \Rightarrow P(X)=\frac{2}{6}=\frac{1}{3}$$

Also $\frac{P(X\cap Y)}{P(Y)}=\frac{1}{3} \Rightarrow P(Y)=\frac{3}{6}=\frac{1}{2}$

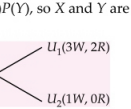
$P(X\cup Y)=P(X)+P(Y)-P(X\cap Y)$

$$=\frac{1}{2}+\frac{1}{3}-\frac{1}{6}=\frac{1}{2}+\frac{1}{6}=\frac{4}{6}=\frac{2}{3}$$

$P(X)P(Y)=\frac{1}{2}\cdot\frac{1}{3}$. Also $P(X\cap Y)=\frac{1}{6}$

As $P(X\cap Y)=P(X)P(Y)$, so X and Y are independent.

62-63 :



62. (b) : $P(W)=P(H\cap W)+P(T\cap W)$

W : White, T : Tail, H : Head

$$=\frac{1}{2}\left\{\frac{3}{5}\cdot 1+\frac{2}{5}\cdot\frac{1}{2}\right\}+\frac{1}{2}\left\{\frac{{}^{3}C_2}{{}^{5}C_2}\cdot 1+\frac{{}^{2}C_2}{{}^{5}C_2}\cdot\frac{1}{3}+\frac{{}^{3}C_1\cdot{}^{2}C_1}{{}^{5}C_2}\cdot\frac{2}{3}\right\}$$

$$=\frac{1}{2}\left\{\frac{4}{5}\right\}+\frac{1}{2}\left\{\frac{3}{10}+\frac{1}{10}\cdot\frac{1}{3}+\frac{3}{5}\cdot\frac{2}{3}\right\}$$

$$=\frac{4}{10}+\frac{1}{2}\left\{\frac{3}{10}+\frac{1}{30}+\frac{2}{5}\right\}$$

$$=\frac{2}{5}+\frac{3}{20}+\frac{1}{60}+\frac{1}{5}=\frac{24+9+1+12}{60}=\frac{46}{60}=\frac{23}{30}.$$

63. (d) : $P\left(\frac{H}{W}\right)=\frac{P(W/H)\cdot P(H)}{P(W/T)P(T)+P(W/H)P(H)}$

$$= \frac{\frac{1}{2}\left\{\frac{3}{5}.1+\frac{2}{5}.\frac{1}{2}\right\}}{\frac{23}{30}} = \frac{\frac{1}{2}.\frac{4}{5}}{\frac{23}{30}} = \frac{\frac{2}{5}}{\frac{23}{30}} = \frac{12}{23}.$$

64. (a, d) : $P(E \cap F) = P(E)\cdot P(F)$

$P(E \cap F') + P(E' \cap F) = \frac{11}{25}$ and $P(E' \cap F') = \frac{2}{25}$

Let $P(E) = x$, $P(F) = y$, then we have

$x(1-y) + y(1-x) = \frac{11}{25}$

$x + y - 2xy = \frac{11}{25}$

Also $(1-x)(1-y) = \frac{2}{25} \Rightarrow 1-(x+y)+xy = \frac{2}{25}$

Solving for $x + y$ and xy gives

$x + y = \frac{7}{5}$ and $xy = \frac{12}{25}$

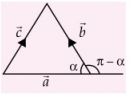
Thus x and y are roots of $t^2 - \frac{7}{5}t + \frac{12}{25} = 0$

$\Rightarrow \left(t - \frac{4}{5}\right)\left(t - \frac{3}{5}\right) = 0 \quad \therefore \quad t = \frac{4}{5}, \frac{3}{5}$

Thus $P(E) = \frac{4}{5}$ and $P(F) = \frac{3}{5}$ or $P(E) = \frac{3}{5}$ and $P(F) = \frac{4}{5}$.

65. (A)-q, (B)-p, (C)-s, (D)-t

(A)



$\cos(\pi-\alpha) = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{-1+3}{\sqrt{1+3}\sqrt{1+3}} = \frac{2}{4} = \frac{1}{2}$

$\cos\alpha = -\frac{1}{2} \therefore \alpha = \frac{2\pi}{3}$

(B) Using leibnitz rule

$f(b) - 3b = -2b \ \therefore \ f(b) = b$

(C) $I = \int_{7/6}^{5/6} \sec(\pi x)dx = \left(\frac{\ln|\sec\pi x + \tan\pi x|}{\pi}\right)_{7/6}^{5/6}$

$= \frac{\ln\left|-\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right| - \ln\left|-\frac{2}{\sqrt{3}} + \frac{1}{\sqrt{3}}\right|}{\pi} = \frac{\ln 3}{\pi}$

Now $\frac{\pi^2}{\ln 3} I = \frac{\pi^2}{\ln 3}.\frac{\ln 3}{\pi} = \pi$

(D) Let $\omega = \frac{1}{1-z} \Rightarrow z = 1 - \frac{1}{\omega}$

$|z| = 1 \ \Rightarrow \left|1 - \frac{1}{\omega}\right| = 1 \ \Rightarrow |1-\omega| = |\omega|$.

Thus the locus of ω is the perpendicular bisector of line segment joining 0 and 1.

Maximum arg ω approaches $\pi/2$ but doesn't attain.

66. (A)-p, (B)-t, (C)-r, (D)-r

(A) 1st solution

$$\mathrm{Re}\left(\frac{2i(x+iy)}{1-(x^2-y^2+2ixy)}\right) = \mathrm{Re}\left(\frac{-2y+2ix}{2y(y-ix)}\right)$$

$$= \mathrm{Re}\left(-\frac{1}{y}\right) = -\frac{1}{y}$$

As $-1 \le y \le 1$, we have the solution set given by $(-\infty, -1) \cup (1, \infty)$

2nd solution : $z = e^{i\theta}$

we have $\frac{2iz}{1-z^2} = \frac{2i}{(1/z)-z} = \frac{2i}{e^{-i\theta}-e^{i\theta}} = \frac{2i}{-2i\sin\theta} = \frac{-1}{\sin\theta}$

$$\mathrm{Re}\left(-\frac{1}{\sin\theta}\right)$$

And now the same solution can be obtained.

(B) To find domain we have $-1 \le \frac{8\cdot 3^{x-2}}{1-3^{2x-2}} \le 1$

Let $u = 3^x$ to obtain $-1 \le \frac{8u}{9-u^2} \le 1$

$|8u| \le (9-u^2)$

$\Rightarrow \quad (u^2+8u-9)(u^2-8u-9) \ge 0$

$\Rightarrow \quad (u+9)(u-1)(u-9)(u+1) \ge 0$

$\Rightarrow \quad (u-1)(u-9) \ge 0$

As u is always + ve $\Rightarrow u \le 1$ or $u \ge 9$

$3^x \le 1$, or $3^x \ge 9 \Rightarrow 3^x \le 3^0$, or $3^x \ge 3^2$

i.e., $x \le 0$, or $x \ge 2 \ \Rightarrow \ x \in (-\infty, 0] \cup [2, \infty)$

(C) $f(\theta) = \begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$

$= \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix} = 2\sec^2\theta,$

applying $R_1 \to R_1 + R_3$.

(D) $f(x) = x^{3/2}(3x-10), \ x \ge 0$

$f'(x) = x^{3/2}\cdot 3 + \frac{3}{2}x^{1/2}(3x-10)$

$= x^{1/2}\left\{3x + \frac{3}{2}(3x-10)\right\}$

$= 3x^{1/2}\left\{x + \frac{3x}{2} - 5\right\} = 3x^{1/2}\left\{\frac{5x}{2} - 5\right\}$

$= \frac{15}{2}x^{1/2}(x-2) \ \ f'(x) \ge 0 \Rightarrow x \ge 2$

SOLVED PAPER 2014

JEE Main

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1. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$, then

(a) $\alpha = -6, \beta = -\frac{1}{2}$ (b) $\alpha = 2, \beta = -\frac{1}{2}$

(c) $\alpha = 2, \beta = \frac{1}{2}$ (d) $\alpha = -6, \beta = \frac{1}{2}$

2. The locus of the foot of the perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is

(a) $(x^2 - y^2)^2 = 6x^2 - 2y^2$
(b) $(x^2 + y^2)^2 = 6x^2 + 2y^2$
(c) $(x^2 + y^2)^2 = 6x^2 - 2y^2$
(d) $(x^2 - y^2)^2 = 6x^2 + 2y^2$

3. Let $f_k(x) = \frac{1}{k}(\sin^k x + \cos^k x)$ where $x \in R$ and $k \geq 1$. Then $f_4(x) - f_6(x)$ equals

(a) $\frac{1}{3}$ (b) $\frac{1}{4}$ (c) $\frac{1}{12}$ (d) $\frac{1}{6}$

4. If $X = \{4^n - 3n - 1, n \in N\}$ and $Y = \{9(n-1) : n \in N\}$, where N is the set of natural numbers, then $X \cup Y$ is equal to

(a) $Y - X$ (b) X (c) Y (d) N

5. If A is an 3×3 non-singular matrix such that $AA' = A'A$ and $B = A^{-1}A'$, then BB' equals

(a) I (b) B^{-1}
(c) $(B^{-1})'$ (d) $I + B$

6. The integral $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$ is equal to

(a) $xe^{x+\frac{1}{x}} + c$ (b) $(x+1)e^{x+\frac{1}{x}} + c$

(c) $-xe^{x+\frac{1}{x}} + c$ (d) $(x-1)e^{x+\frac{1}{x}} + c$

7. The area of the region described by $A = \{(x, y) : x^2 + y^2 \leq 1 \text{ and } y^2 \leq 1 - x\}$ is

(a) $\frac{\pi}{2} - \frac{4}{3}$ (b) $\frac{\pi}{2} - \frac{2}{3}$

(c) $\frac{\pi}{2} + \frac{2}{3}$ (d) $\frac{\pi}{2} + \frac{4}{3}$

8. The image of the line $\frac{x-1}{3} = \frac{y-3}{1} = \frac{z-4}{-5}$ in the plane $2x - y + z + 3 = 0$ is the line

(a) $\frac{x+3}{-3} = \frac{y-5}{-1} = \frac{z+2}{5}$

(b) $\frac{x-3}{3} = \frac{y+5}{1} = \frac{z-2}{-5}$

(c) $\frac{x-3}{-3} = \frac{y+5}{-1} = \frac{z-2}{5}$

(d) $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$

9. The variance of first 50 even natural numbers is

(a) 833 (b) 437 (c) $\frac{437}{4}$ (d) $\frac{833}{4}$

10. If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z + \frac{1}{2}\right|$

(a) lies in the interval (1, 2)

(b) is strictly greater than $\frac{5}{2}$

(c) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

(d) is equal to $\frac{5}{2}$

11. Three positive numbers form an increasing G.P. If the middle term in this G.P. is doubled, the new numbers are in A.P. Then the common ratio of the G.P. is

(a) $3 + \sqrt{2}$ (b) $2 - \sqrt{3}$ (c) $2 + \sqrt{3}$ (d) $\sqrt{2} + \sqrt{3}$

12. If the coeffients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to

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(a) $\left(14, \frac{251}{3}\right)$ (b) $\left(14, \frac{272}{3}\right)$

(c) $\left(16, \frac{272}{3}\right)$ (d) $\left(16, \frac{251}{3}\right)$

13. Let a, b, c and d be non-zero numbers. If the point of intersection of the lines $4ax + 2ay + c = 0$ and $5bx + 2by + d = 0$ lies in the fourth quadrant and is equidistant from the two axes, then

(a) $2bc + 3ad = 0$ (b) $3bc - 2ad = 0$

(c) $3bc + 2ad = 0$ (d) $2bc - 3ad = 0$

14. If $[\vec{a} \times \vec{b} \quad \vec{b} \times \vec{c} \quad \vec{c} \times \vec{a}] = \lambda [\vec{a}\ \vec{b}\ \vec{c}]^2$, then λ is equal to

(a) 3 (b) 0 (c) 1 (d) 2

15. Let A and B be two events such that

$P(\overline{A \cup B}) = \frac{1}{6}$, $P(A \cap B) = \frac{1}{4}$ and $P(\bar{A}) = \frac{1}{4}$, where $\bar{A}$ stands for the complement of the event A. Then the events A and B are

(a) equally likely but not independent

(b) independent but not equally likely

(c) independent and equally likely

(d) mutually exclusive and independent

16. Let PS be the median of the triangle whose vertices $P(2, 2)$, $Q(6, -1)$ and $R(7, 3)$. The equation of the line passing through $(1, -1)$ and parallel to PS is

(a) $2x + 9y + 7 = 0$ (b) $4x + 7y + 3 = 0$

(c) $2x - 9y - 11 = 0$ (d) $4x - 7y - 11 = 0$

17. $\lim\limits_{x \to 0} \frac{\sin(\pi \cos^2 x)}{x^2}$ is equal to

(a) 1 (b) $-\pi$ (c) π (d) $\pi/2$

18. Let α and β be the roots of the equation $px^2 + qx + r = 0$. If p, q, r are in A.P. and $\frac{1}{\alpha} + \frac{1}{\beta} = 4$, then the value of $|\alpha - \beta|$ is

(a) $\frac{2\sqrt{17}}{9}$ (b) $\frac{\sqrt{34}}{9}$

(c) $\frac{2\sqrt{13}}{9}$ (d) $\frac{\sqrt{61}}{9}$

19. A bird is sitting on the top of a vertical pole 20 m high and its elevation from a point O on the ground is 45°. It flies off horizontally straight away from the point O. After one second, the elevation of the bird from O is reduced to 30°. Then the speed (in m/s) of the bird is

(a) $40(\sqrt{3} - \sqrt{2})$ (b) $20\sqrt{2}$

(c) $20(\sqrt{3} - 1)$ (d) $40(\sqrt{2} - 1)$

20. If $a \in R$ and the equation

$-3(x - [x])^2 + 2(x - [x]) + a^2 = 0$

(where $[x]$ denotes the greatest integer ≤ 3) has no integral solution, then all possible values of a lie in the interval

(a) (1, 2) (b) (–2, –1)

(c) $(-\infty, -2) \cup (2, \infty)$ (d) $(-1, 0) \cup (0, 1)$

21. The integral $\int_0^{\pi} \sqrt{1 + 4\sin^2 \frac{x}{2} - 4\sin \frac{x}{2}}\, dx$ equals

(a) $\frac{2\pi}{3} - 4 - 4\sqrt{3}$ (b) $4\sqrt{3} - 4$

(c) $4\sqrt{3} - 4 - \frac{\pi}{3}$ (d) $\pi - 4$

22. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$

(a) $2f'(c) = 3g'(c)$ (b) $f'(c) = g'(c)$

(c) $f'(c) = 2g'(c)$ (d) $2f'(c) = g'(c)$

23. If g is the inverse of a function f and $f'(x) = \frac{1}{1 + x^5}$, then $g'(x)$ is equal to

(a) $5x^4$ (b) $\frac{1}{1 + \{g(x)\}^5}$

(c) $1 + \{g(x)\}^5$ (d) $1 + x^5$

24. If $(10)^9 + 2(11)^1 (10)^8 + 3(11)^2 (10)^7 + \ldots + 10(11)^9 = k(10)^9$, then k is equal to

(a) $\frac{441}{100}$ (b) 100 (c) 110 (d) $\frac{121}{100}$

25. If $\alpha, \beta \neq 0$ and $f(n) = \alpha^n + \beta^n$ and

$$\begin{vmatrix} 3 & 1 + f(1) & 1 + f(2) \\ 1 + f(1) & 1 + f(2) & 1 + f(3) \\ 1 + f(2) & 1 + f(3) & 1 + f(4) \end{vmatrix} = K(1 - \alpha)^2 (1 - \beta)^2 (\alpha - \beta)^2,$$

then K is equal to

(a) $\frac{1}{\alpha\beta}$ (b) 1 (c) –1 (d) $\alpha\beta$

26. The slope of the line touching both the parabolas $y^2 = 4x$ and $x^2 = -32y$ is

(a) $\frac{3}{2}$ (b) $\frac{1}{8}$ (c) $\frac{2}{3}$ (d) $\frac{1}{2}$

27. The statement $\sim (p \leftrightarrow \sim q)$ is

(a) equivalent to $\sim p \leftrightarrow q$

(b) a tautology

(c) a fallacy

(d) equivalent to $p \leftrightarrow q$

28. Let the population of rabbits surviving at a time t be governed by the differential equation

$\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If $p(0) = 100$ then $p(t)$ equals

(a) $300 - 200\,e^{-t/2}$ (b) $600 - 500\,e^{t/2}$

(c) $400 - 300\,e^{-t/2}$ (d) $400 - 300\,e^{t/2}$

29. Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at $(0, y)$, passing through origin and touching the circle C externally then the radius of T is equal to

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{1}{2}$ (c) $\frac{1}{4}$ (d) $\frac{\sqrt{3}}{\sqrt{2}}$

30. The angle between the lines whose direction cosines satisfy the equations $l + m + n = 0$ and $l^2 + m^2 + n^2$ is

(a) $\frac{\pi}{4}$ (b) $\frac{\pi}{6}$ (c) $\frac{\pi}{2}$ (d) $\frac{\pi}{3}$

SOLUTIONS

1. (b): We have $f(x) = \alpha \log|x| + \beta x^2 + x$

$$f'(x) = \frac{\alpha}{x} + 2\beta x + 1 = \frac{2\beta x^2 + x + \alpha}{x}$$

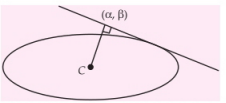
-1 and 2 are the roots of $2\beta x^2 + x + \alpha = 0$

Hence $-\frac{1}{2\beta} = -1 + 2 \Rightarrow -\frac{1}{2\beta} = 1 \therefore \beta = -\frac{1}{2}$

$$\frac{\alpha}{2\beta} = (-1)(2) \Rightarrow \frac{\alpha}{2\beta} = -2 \therefore \alpha = 2$$

[Topic : Maxima and Minima]
Rating : Easy

2. (b): 1st solution:



The equation of tangent to the ellipse $\frac{x^2}{6} + \frac{y^2}{2} = 1$ is

$$\frac{x\cos\theta}{\sqrt{6}} + \frac{y\sin\theta}{\sqrt{2}} = 1 \quad \ldots (1)$$

The equation to the tangent is also given by

$$y - \beta = -\frac{\alpha}{\beta}(x - \alpha) \quad i.e., \alpha x + \beta y = \alpha^2 + \beta^2 \quad \ldots (2)$$

Comparing (1) and (2), we get

$$\frac{\cos\theta}{\sqrt{6}\alpha} = \frac{\sin\theta}{\sqrt{2}\beta} = \frac{1}{\alpha^2 + \beta^2} = \frac{\sqrt{\cos^2\theta + \sin^2\theta}}{\sqrt{6\alpha^2 + 2\beta^2}}$$

$\Rightarrow 6\alpha^2 + 2\beta^2 = (\alpha^2 + \beta^2)^2$

$\therefore$ Locus of (α, β) is $(x^2 + y^2)^2 = 6x^2 + 2y^2$

2nd solution : The equation of the tangent is

$$y = mx \pm \sqrt{6m^2 + 2}$$

and slope m is given by $m = -\frac{\alpha}{\beta}$

Also, (α, β) lies on the tangent

$$\Rightarrow \beta = m\alpha \pm \sqrt{6m^2 + 2} \Rightarrow \beta = \left(-\frac{\alpha}{\beta}\right)\alpha \pm \sqrt{\frac{6\alpha^2}{\beta^2} + 2}$$

$$\Rightarrow \frac{\alpha^2 + \beta^2}{\beta} = \pm\sqrt{\frac{6\alpha^2 + 2\beta^2}{\beta^2}} \Rightarrow (\alpha^2 + \beta^2)^2 = 6\alpha^2 + 2\beta^2$$

Then the locus is $(x^2 + y^2)^2 = 6x^2 + 2y^2$.

[Topic : Ellipse]
Rating : Doable

3. (c): $f_4(x) - f_6(x) = \frac{1}{4}(\sin^4 x + \cos^4 x) - \frac{1}{6}(\cos^6 x + \sin^6 x)$

$= \frac{1}{4}\{(\sin^2 x + \cos^2 x)^2 - 2\sin^2 x\cos^2 x\} - \frac{1}{6}\{(\cos^2 x + \sin^2 x)^3 - 3\cos^2 x\sin^2 x\}$

$= \frac{1}{4}(1 - 2\sin^2 x\cos^2 x) - \frac{1}{6}(1 - 3\sin^2 x\cos^2 x)$

$= \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$

Remark : As the given expression is independent of x, as suggested by choices, one can simply put a convenient value to obtain the result at $x = 0$.

Hence $f_4(0) - f_6(0) = \frac{1}{4} - \frac{1}{6} = \frac{1}{12}$ etc.

[Topic : Trigonometric Expressions]
Rating : Easy

4. (c): $X = \{4^n - 3n - 1\}$, $Y = \{9(n-1) : n \in N\}$

$4^n - 3n - 1 = (1 + 3)^n - 3n - 1$

$= (1 + {}^nC_1\,3 + {}^nC_2\,3^2 + \ldots + 3^n) - 3n - 1$

$= (1 + 3n) + 9({}^nC_2 + \ldots + 3^{n-2}) - 3n - 1$

$= 9({}^nC_2 + \ldots + 3^{n-2}),\ n \geq 2$

$= 9\lambda$.

Also for $n = 1$, $4^n - 3n - 1 = 0$, a multiple of 9

Every element in X is a multiple of 9. But Y contains all multiples of 9. Hence $X \cup Y = Y$.

[Topic : Set theory, Binomial Theorem]
Rating : Difficult

5. (a): $BB^T = (A^{-1}A^T)(A^{-1}A^T)^T$

$= (A^{-1}A^T)((A^T)^T(A^{-1})^T) = (A^{-1}A^T)(A(A^{-1})^T)$

$= A^{-1}(A^TA)(A^T)^{-1} = A^{-1}(AA^T)(A^T)^{-1}$

$= (A^{-1}A)(A^T)(A^T)^{-1} = I \cdot I = I$

Recall that $(AB)^T = B^T A^T$ and that the matrix multiplication is associative.

[Topic : Matrices]
Rating : Easy

6. (a) : $\int \left(1 + x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$

$= \int 1 \cdot e^{x+\frac{1}{x}}\, dx + \int \left(x - \frac{1}{x}\right) e^{x+\frac{1}{x}}\, dx$

$= \int e^{x+\frac{1}{x}}\, dx + \int x\left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}}\, dx$

$= \int e^{x+\frac{1}{x}}\, dx + \left[x \int \left(1 - \frac{1}{x^2}\right) e^{x+\frac{1}{x}}\, dx - \int 1 \cdot e^{x+\frac{1}{x}} dx\right]$

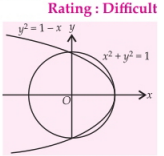
(Integration by parts)

$= \int e^{x+\frac{1}{x}}\, dx + x\, e^{x+\frac{1}{x}} - \int e^{x+\frac{1}{x}}\, dx = x\, e^{x+\frac{1}{x}} + c$

c being an arbitrary constant.

[Topic : Indefinite Integration]
Rating : Difficult

7. (d): 1st solution :



$$\text{Area} = \frac{\pi}{2} + 2\int_0^1 \sqrt{1-x}\, dx$$

$$= \frac{\pi}{2} + 2 \cdot \frac{(1-x)^{3/2}}{3/2}\Big|_0^1$$

$$= \frac{\pi}{2} + \frac{4}{3}[(1-x)^{3/2}]_0^1 = \frac{\pi}{2} + \frac{4}{3}$$

2nd solution: $\text{Area} = \frac{\pi}{2} + 2\int_0^1 x\, dy$

$$= \frac{\pi}{2} + 2\int_0^1 (1-y^2)dy = \frac{\pi}{2} + 2\left(y - \frac{y^3}{3}\right)\Big|_0^1$$

$$= \frac{\pi}{2} + 2\left(1 - \frac{1}{3}\right) = \frac{\pi}{2} + \frac{4}{3}$$

[Topic : Area]
Rating : Doable

8. (d): The line is parallel to the plane.

Image of (1, 3, 4) in the plane $2x - y + z + 3 = 0$ is given by

$$\frac{\alpha - 1}{2} = \frac{\beta - 3}{-1} = \frac{\gamma - 4}{1} = \frac{-2(2-3+4+3)}{\sqrt{2^2+1^2+1^2}} = \frac{-2 \times 6}{6} = -2$$

$\therefore \quad (\alpha, \beta, \gamma) \equiv (-3, 5, 2)$

Thus the required line is $\frac{x+3}{3} = \frac{y-5}{1} = \frac{z-2}{-5}$

[Topic : 3D Geometry]
Topic : Doable

9. (a) : We have $\alpha^2 = \frac{\Sigma x_i^2}{h} - \left(\frac{\Sigma x_i}{h}\right)^2$

$\Sigma x_i^2 = 2^2 + 4^2 + ... + 100^2 = 4\{1^2 + 2^2 + ... + 50^2\}$

$$= \frac{4 \cdot 50 \times 51 \times 101}{6}$$

$\therefore \quad \frac{\Sigma x_i^2}{50} = \frac{4 \times 51 \times 101}{6} = 3434$

$\left(\frac{\Sigma x_i}{50}\right)^2 = \left(\frac{2 \cdot 50 \cdot 51}{2 \times 50}\right)^2 = 2601$

$\therefore \quad \alpha^2 = 3434 - 2601 = 833$

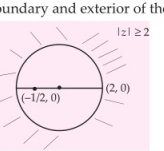
[Topic : Statistics]
Rating : Easy

10. (a) : 1st solution : $\left|z + \frac{1}{2}\right| \geq \left||z| - \frac{1}{2}\right|$

As $|z| \geq 2$ the minimum value of the expression occurs when $|z| = 2$

Thus $\left|z + \frac{1}{2}\right|_{\min} = \frac{3}{2}$

2nd solution : Geometrically $|z| = 2$ is a circle and $|z| \geq 2$ is the boundary and exterior of the circle.



The minimum distance between z and point (–1/2, 0) is realised at (2, 0) and is 3/2.

[Topic : Complex Numbers]
Rating : Doable

11. (c) : Let the number be a, ar, ar^2, we have

$2|2ar| = a + ar^2 \quad \Rightarrow \quad 4r = r^2 + 1 \Rightarrow r^2 - 4r + 1 = 0$

$\Rightarrow \quad r = \frac{4 \pm \sqrt{12}}{2} = \frac{4 \pm 2\sqrt{3}}{2} = 2 \pm \sqrt{3}$

$\therefore \quad r = 2 + \sqrt{3}$, as number is positive.

[Topic : Sequences and Series]
Rating : Easy

12. (c) : The given expansion is

$(1 - 2x)^{18} + ax(1 - 2x)^{18} + bx^2(1 - 2x)^{18}$

The coefficient of x^3 is ,

$(-2)^3 \cdot {}^{18}C_3 + a(-2)^2 \cdot {}^{18}C_2 + b(-2) \cdot {}^{18}C_1 = 0 \qquad ...(1)$

The coefficient of x^4 is,

$(-2)^4 \cdot {}^{18}C_4 + a(-2)^3 \cdot {}^{18}C_3 + b(-2)^2 \cdot {}^{18}C_2 = 0 \qquad ...(2)$

From (1) and (2), we get

$153a - 9b = 1632 \qquad ...(3)$

$3b - 32a = -240$ ...(4)

Solving (3) and (4), we get $a = 16,\ b = \frac{272}{3}$

[Topic : Binomial Theorem]
Rating : Doable

13. (b): Let the point of intersection be $(\alpha, -\alpha)$. Then
$4a\alpha - 2a\alpha + c = 0$
$5b\alpha - 2b\alpha + d = 0$
We have,

$$-\frac{c}{2a} = -\frac{d}{3b} \Rightarrow 3bc - 2ad = 0$$

[Topic : Straight Lines]
Rating : Doable]

14. (c): $\left[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}\right] = (\vec{a}\times\vec{b})\{(\vec{b}\times\vec{c})\times(\vec{c}\times\vec{a})\}$

$= (\vec{a}\times\vec{b})\{(\vec{b}\times\vec{c}\cdot\vec{a})\vec{c} - (\vec{b}\times\vec{c}\cdot\vec{c})\vec{a}\}$

$= (\vec{a}\times\vec{b})[\vec{a}\ \ \vec{b}\ \ \vec{c}]\vec{c} = [\vec{a}\ \vec{b}\ \vec{c}][\vec{a}\ \vec{b}\ \vec{c}] = [\vec{a}\ \vec{b}\ \vec{c}]^2$

$\therefore$ On comparison, $\lambda = 1$

[Topic : Vectors]
Rating : Doable

15. (b): $P(A \cup B) = 1 - P(\overline{A \cup B}) = 1 - \frac{1}{6} = \frac{5}{6}$

We have $P(\bar{A}) = \frac{1}{4} \Rightarrow P(A) = \frac{3}{4}$

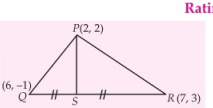
Again, $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

$\Rightarrow \frac{5}{6} = \frac{3}{4} + P(B) - \frac{1}{4} \quad \therefore \ P(B) = \frac{1}{3}$

Now, $P(A \cap B) = P(A)\,P(B)$ is seem to be true.
Thus A and B are independent.
As $P(A) \neq P(B)$, A and B are not equally likely.

[Topic : Probability]
Rating : Doable

16. (a):



S is $\left(\frac{13}{2}, 1\right)$

Slope of $PS = -2/9$
The equation of line passing through $(1, -1)$ and parallel to PS is $y + 1 = -\frac{2}{9}(x - 1) \Rightarrow 2x + 9y + 9 - 2 = 0$

$\therefore \quad 2x + 9y + 7 = 0$

[Topic : Straight Lines]
Rating : Easy

17. (c): $\lim_{x\to 0} \frac{\sin(\pi\cos^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin\{\pi - \pi(\cos^2 x)\}}{x^2}$

$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{x^2} = \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{(\pi\sin^2 x)} \cdot \frac{\pi\sin^2 x}{x^2}$

$= \lim_{x\to 0} \frac{\sin(\pi\sin^2 x)}{\pi\sin^2 x} \cdot \pi \cdot \left(\frac{\sin x}{x}\right)^2 = 1 \cdot \pi \cdot 1 = \pi$

[Topic : Limits]
Rating : Doable

18. (c): We have $\frac{1}{\alpha} + \frac{1}{\beta} = 4$ and $2q = p + r$

Also, $-2(\alpha + \beta) = \alpha\beta + 1$

$\Rightarrow \quad -2\left(\frac{1}{\alpha} + \frac{1}{\beta}\right) = 1 + \frac{1}{\alpha\beta} \Rightarrow \frac{1}{\alpha\beta} = -9$

The equation having roots $\frac{1}{\alpha}$ and $\frac{1}{\beta}$ is

$x^2 - 4x - 9 = 0$

The equation having roots α and β is

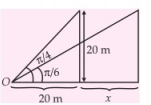
$9x^2 + 4x - 1 = 0$

$$\alpha, \beta = \frac{-4 \pm \sqrt{16 + 36}}{2 \times 9} = \frac{-4 \pm 2\sqrt{13}}{2 \times 9} = \frac{-2 \pm \sqrt{13}}{9}$$

$\therefore \ |\alpha - \beta| = \frac{2\sqrt{13}}{9}$

[Topic : Quadratic Equations]
Rating : Doable

19. (c):



We have $\tan 30° = \frac{20}{20 + x} = \frac{1}{\sqrt{3}}$

$\Rightarrow \quad 20 + x = 20\sqrt{3} \Rightarrow x = 20(\sqrt{3} - 1)$

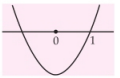
The speed of bird is $20(\sqrt{3} - 1)$ m/s

[Topic : Heights and Distances]
Rating : Easy

20. (d): Let $\{x\} = t$, so $0 \le t < 1$, we have
$3t^2 - 2t - a^2 = 0$
$D = 4 + 12a^2 > 0$
Then the equation has discriminant as positive. Assume that the roots are between 0 and 1, we have to ensure that there is no integral root, that is $t \neq 0$.
When $t = 0 \Rightarrow a^2 = 0 \quad \therefore \quad a = 0$

Product of roots $= -\frac{a^2}{3} < 0$

Thus one root is positive and the other is negative.
The condition that the root is strictly between 0 and 1 is



$f(0)\,f(1) < 0 \Rightarrow -a^2(1-a^2) < 0$

$\Rightarrow a^2 - 1 < 0 \quad \Rightarrow a^2 < 1. \quad \therefore \quad a \in (-1, 1)$

For integral roots, $a = 0$

$\therefore$ The set of all possible values of a is $(-1, 0) \cup (0, 1)$.

Remark : The question assumes that the equation does have a solution. Otherwise no answer is correct.

[Topic : Quadratic Equations]
Rating : Difficult

21. (c) : $\displaystyle\int_0^{\pi} \sqrt{1 + 4\sin^2\frac{x}{2} - 4\sin\frac{x}{2}}\,dx$

$$= \int_0^{\pi} \sqrt{\left(1 - 2\sin\frac{x}{2}\right)^2}\,dx = \int_0^{\pi} \left|1 - 2\sin\frac{x}{2}\right| dx$$

$$= \int_0^{\pi/3} \left(1 - 2\sin\frac{x}{2}\right) dx + \int_{\pi/3}^{\pi} \left(2\sin\frac{x}{2} - 1\right) dx$$

$$= \left(x + 4\cos\frac{x}{2}\right)\Bigg|_0^{\pi/3} + \left(-4\cos\frac{x}{2} - x\right)\Bigg|_{\pi/3}^{\pi}$$

$$= -\frac{\pi}{3} + 8\cdot\frac{\sqrt{3}}{2} - 4 = 4\sqrt{3} - 4 - \frac{\pi}{3}$$

[Topic : Definite Integration]
Rating : Doable

22. (c) : $\dfrac{f'(c)}{g'(c)} = \dfrac{f(1) - f(0)}{g(1) - g(0)}$

$$= \frac{6-2}{2-0} = 2$$

$\Rightarrow f'(c) = 2g'(c)$

[Topic : Differentiability]
Rating : Difficult

23. (c) : $f(g(x)) = x$

$f'(g(x))g'(x) = 1$

$\Rightarrow \quad g'(x) = \dfrac{1}{f'(g(x))} \quad \Rightarrow \quad g'(x) = 1 + \{g(x)\}^5$

[Topic : Differentiability]
Rating : Doable

24. (b): Let $P = 10^9 + 2\cdot 11\cdot 10^8 + \ldots + 10\cdot 11^9$

we have $\dfrac{11}{10}P = 11\cdot 10^8 + \ldots + 9\cdot 11^9 + 11^{10}$

on subtracting, we get

$$\Rightarrow \frac{1}{10}P = 11^{10} - [10^9 + 11^1\cdot 10^8 + 11^2\cdot 10^7 + \ldots + 11^9]$$

$$= 11^{10} - 10^9\left\{\frac{\left(\frac{11}{10}\right)^{10} - 1}{\frac{11}{10} - 1}\right\} = 11^{10} - 11^{10} + 10^{10}$$

$\therefore \quad P = 10^{11} = (100)\cdot 10^9$

On comparison, $k = 100$

[Topic : Sequences and Series]
Rating : Difficult

25. (b):
$$\begin{vmatrix} 3 & 1+\alpha+\beta & 1+\alpha^2+\beta^2 \\ 1+\alpha+\beta & 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 \\ 1+\alpha^2+\beta^2 & 1+\alpha^3+\beta^3 & 1+\alpha^4+\beta^4 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \beta \\ 1 & \alpha^2 & \beta^2 \end{vmatrix} \begin{vmatrix} 1 & 1 & 1 \\ 1 & \alpha & \alpha^2 \\ 1 & \beta & \beta^2 \end{vmatrix}$$

By multiplication of determinant

$= [(1-\alpha)(1-\beta)(\beta-\alpha)]^2$

On comparison, $K = 1$

[Topic : Determinants]
Rating : Difficult

26. (d): The equation of tangent at $P(t^2, 2t)$ is

$x - yt + t^2 = 0$

$x^2 + 32y = 0$ is the given curve.

Eliminating y, $x^2 + \dfrac{32}{t}x + 32t = 0$

For tangent, discriminant = 0

$$\Rightarrow \left(\frac{32}{t}\right)^2 - 4(32t) = 0$$

$$\Rightarrow \frac{32}{t^2} - 4t = 0$$

$\Rightarrow t^3 = 8 \quad \therefore \quad t = 2$

Slope of tangent is 1/2.

[Topic : Parabola]
Rating : Difficult

27. (d): See the following truth table.

p	q	$\sim q$	$p \leftrightarrow \sim q$	$\sim (p \leftrightarrow \sim q)$	$p \leftrightarrow q$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

As the truth table matches, we have the statement

$\sim (p \leftrightarrow \sim q)$ is equivalent to $p \leftrightarrow q$

[Topic : Mathematical Reasoning]
Rating : Doable

28. (d): $\frac{dp}{dt} = \frac{1}{2}p(t) - 200$

$$\frac{dp}{p-400} = \frac{1}{2}dt$$

Integrating, we get, $\ln|p-400| = \frac{1}{2}t + c$

$t = 0,\ p = 100 \Rightarrow \ln 300 = c$

Again, $\ln\left(\frac{p-400}{300}\right) = \frac{t}{2} \Rightarrow |p-400| = 300e^{t/2}$

$\therefore \quad 400 - p = 300e^{t/2} \quad (\because p < 400)$

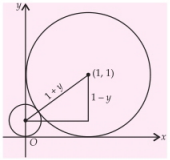
$\therefore \quad p = 400 - 300e^{t/2}$

[Topic Differential Calculus]
Rating : Easy

29. (c) : We have by Pythagoras theorem

$(1+y)^2 = (1-y)^2 + 1$

$\Rightarrow \quad 4y = 1 \qquad \therefore \quad y = 1/4$



[Topic : Circles]
Rating : Doable

30. (d): As $l = -m - n$. We have $l^2 = m^2 + n^2$ gives

$m^2 + n^2 = (m+n)^2 \Rightarrow 2mn = 0 \Rightarrow mn = 0$

So, the d.r.'s is $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0\right)$ or $\left(-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

$$\cos\theta = \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} + 0 + 0 \Rightarrow \quad \cos\theta = \frac{1}{2} \quad \Rightarrow \quad \theta = \frac{\pi}{3}$$

[Topic : Three Dimensional Geometry]
Rating : Doable

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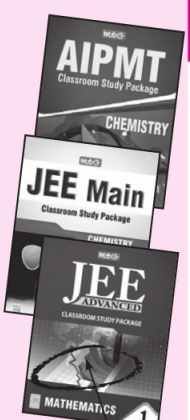

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IMO Problems From TEAM SELECTION TESTS

* ALOK KUMAR, B.Tech, IIT Kanpur

PROBLEMS

1. Two brothers had a flock of sheep. They sold the flock and got as many rupees for every sheep as was the number of the sheep in the flock. The money was shared in the following way : First, the elder brother took ten rupees from the cash, then the younger brother took ten rupees, after which the elder brother took ten rupees again, and so on. Finally, it turned out that at the last stage when it was the younger brother's turn to take money there remained less than ten rupees. Therefore the younger brother took the rest of the money and the elder brother gave him his knife for the sharing to be fair. Find the cost of the knife.

2. Find the real points (x, y) satisfying
$3x^2 + 3y^2 - 4xy + 10x - 10y + 10 = 0$.

3. For any $x \in R$ and $n \in N$, define $x^{(0)} = 1$ and $x^{(n)} = x(x - 1)..... (x - n + 1)$. Show that
$$(x+y)^{(n)} = \sum_{k=0}^{n} \binom{x}{k} x^{(k)} y^{(n-k)}$$

4. Show that the centroid of a triangle coincides with that of the triangle formed by joining the midpoints of its sides.

5. In an acute-angled triangle ABC, $\angle A = 30°$, O is the orthocentre and M is the mid point of BC. On the line OM, T is the point such that $OM = MT$. Show that $AT = 2BC$.

6. In a triangle ABC, and A is twice angle B. Show that $a^2 = b(b + c)$.

7. A rhombus has half the area of the square with the same side-length. Find the ratio of the long diagonals to the short one.

8. Find all continuous functions $f : R \to R$ such that
$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2 .$$

9. Find all integral solutions of $x^3 + 5y^3 + 25z^3 - 15xyz = 0$.

10. Find all positive integers x, y satisfying
$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}}.$$

11. Solve the following equation in natural numbers:
$x^2 + y^2 = 1997(x - y)$.

12. Find all finite sets A of distinct non-negative real numbers for which:
    (a) the set A contains at least 4 numbers;
    (b) for any 4 distinct numbers $a, b, c, d \in A$, the number $ab + cd \in A$.

13. Let n be a natural number such that $n \geq 2$. Show that
$$\frac{1}{n+1}\left(1 + \frac{1}{3} + ... + \frac{1}{2n-1}\right) > \frac{1}{n}\left(\frac{1}{2} + \frac{1}{4} + ... + \frac{1}{2n}\right).$$

14. Let m be a positive integer. Define the sequence $\{a_n\}_{n \geq 0}$ by $a_0 = 0$, $a_1 = m$ and $a_{n+1} = m^2 a_n - a_{n-1}$ for $n \geq 1$. Prove that an ordered pair (a, b) of non-negative integers, with $a \leq b$, is a solution of the equation $\frac{a^2 + b^2}{ab + 1} = m^2$ if and only if $(a, b) = (a_n, a_{n+1})$ for some $n \geq 0$.

15. Determine all the positive integers $n \geq 3$, such that 2^{2000} is divisible by $1 + \binom{n}{1} + \binom{n}{2} + \binom{n}{3}$.

16. Find all real numbers x such that $x\lfloor x\lfloor x\lfloor x \rfloor \rfloor \rfloor = 88$

17. A point A is given outside of a circle k in the plane. Show that the diagonals of any trapezoid inscribed in k, the extensions of whose non-parallel sides meet at A, meet at a point independent of the choice of trapezoid.

18. We are given a 3×3 table with integer entries, not all of the same parity. We repeat the following operation: simultaneously replace each entry by the sum of its neighbours (those entries sharing an edge with the given entry). Regardless of

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the starting position, will we always eventually obtain a table with all entries even? With all entries odd?

19. Let ABC be a triangle and P, Q, R points on AB, BC, CA, respectively. The points A', B', C' on RP, PQ, QR, respectively, have the property that $AB \| A'B'$, $BC \| B'C'$, $CA \| C'A'$. Show that $\frac{AB}{A'B'} = \frac{[PQR]}{[A'B'C']}$.

20. (a) Show that the product of two numbers of the form $a^2 + 3b^2$ is again of that form.

(b) If an integer n is such that $7n$ is of the form $a^2 + 3b^2$, prove that n is also of that form.

21. Suppose a, b are natural numbers such that

$p = \frac{b}{4}\sqrt{\frac{2a-b}{2a+b}}$ is a prime number. What is the maximum possible value of p?

22. Let $x, y, z > 1$ and $\frac{1}{x}+\frac{1}{y}+\frac{1}{z} = 2$. Prove that

$\sqrt{x+y+z} \geq \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$

23. Let $c_{n,\,m}$ be the number of permutations of $\{1, ..., n\}$ which can be written as the product of m transpositions of the form $(i, i+1)$ for some $i = 1, ..., n-1$ but not of $m-1$ such transpositions. Prove that for all $n \in N$,

$$\sum_{m=0}^{\infty} c_{n,\,m}\, t^m = \prod_{i=1}^{n}(1 + t + \cdots + t^{i-1})$$

SOLUTIONS

1. Suppose that the number of sheep in the flock is $10a + b$ where $0 \leq b \leq 9$. Then the total cash received is $100a^2 + 20ab + b^2$. Since the money remained, was less than rupees 10, when it was the younger brother's turn, we must have $b^2 = 10t + r$ where t is odd and r is the money he took finally; further $b^2 \in \{0,1,4,16,25,36,49,64,81\}$; therefore it follows that b^2 is either 16 or 36. Hence $r = 6$. Let the cost of the knife be rupees x. Since the sharing became fair, after the elder gave the knife to the younger, we have $10 - x = 6 + x$; thus the cost of the knife is rupees 2.

2. The given equation can be considered as a quadratic in x :

$3x^2 + (10 - 4y)x + (3y^2 - 10y + 10) = 0$

Solving for x, we get

$$x = \frac{1}{6}\left(4y - 10 \pm \sqrt{(10-4y)^2 - 12(3y^2 - 10y + 10)}\right)$$

Since x, y are real $(10-4y)^2 - 12(3y^2 - 10y + 10) \geq 0$ which on simplification gives $(y-1)^2 \leq 0$; *i.e.*, $y = 1$ and therefore $x = -1$.

Alternative solution : L.H.S. of the given equation

$= 3(x^2 + 2x + 1) + 3(y^2 - 2y + 1) + 4(1 + x - y - xy)$

$= 3(1+x)^2 + 3(1-y)^2 + 4(1+x)(1-y)$

$= (1+x)^2 + (1-y)^2 + 2[(1+x)^2 + (1-y)^2 + 2(1+x)(1-y)]$

$= (1+x)^2 + (1-y)^2 + 2[(1+x) + (1-y)]^2$

Since x, y are real, $1 + x = 0 = 1 - y$; *i.e.*, $(x, y) = (-1, 1)$.

3. **Proof by induction :** Result is clearly true for $n = 1$. Assume the relation to be true for all $n \leq m$. To prove the result for $n = m + 1$:

$$\sum_{k=0}^{m+1}\binom{m+1}{k}x^{(k)}y^{(m+1-k)} = \sum_{k=0}^{m+1}\left[\binom{m}{k}+\binom{m}{k-1}\right]x^{(k)}y^{(m+1-k)}$$

$$\left[\text{Note}: \binom{m}{m+1} = \binom{m}{-1} = 0\right]$$

$$= \sum_{k=0}^{m}\binom{m}{k}x^{(k)}(y-1)^{(m-k)}y + \sum_{k=0}^{m+1}\binom{m}{k-1}x\cdot(x-1)^{(k-1)}y^{(m+1-k)}$$

$$= y\sum_{k=0}^{m}\binom{m}{k}x^{(k)}(y-1)^{(m-k)} + x\sum_{l=0}^{m}\binom{m}{l}(x-1)^{(l)}y^{(m-l)}$$

$= y(x+y-1)^{(m)} + x(x-1+y)^{(m)}$

(by induction hypothesis)

$= (x+y)(x+y-1)^{(m)} = (x+y)^{(m+1)}$

Alternative solution : By using Leibnitz's theorem for successive differentiation. If u and v are two functions of a variable t, each being differentiable n times then uv also has n^{th} derivative given as follows :

$$\frac{d^n(uv)}{dt^n} = \sum_{k=0}^{n}\binom{n}{k}\frac{d^k u}{dt^k}\cdot\frac{d^{n-k}v}{dt^{n-k}}$$

Let t^x, $v = t^y$. Then L.H.S. =

$(x+y)(x+y-1)\ldots(x+y-n+1)t^{x+y-n}$

$(x+y)^{(n)}\,t^{x+y-n}$

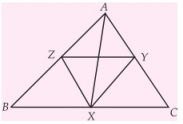
R.H.S = $\sum_{k=0}^{n}\binom{n}{k}x(x-1)\ldots(x-k+1)t^{x-k}\times y(y-1)\ldots(y-(n-k)+1)t^{y-(n-k)}$

$= \left(\sum_{k=0}^{n}\binom{n}{k}x^{(k)}y^{(n-k)}\right)t^{x+y-n}$

So, cancelling t^{x+y-n} we get the required result.

4. Let ABC be a triangle and X, Y, Z be the midpoints of the sides BC, CA, AB respectively. Since $AZXY$ is a parallelogram, its diagonal ZY is bisected by the other diagonal AX; therefore AX is also a median of triangle XYZ. Thus both triangles ABC and XYZ have the same medians and therefore also the same centroid.

Alternative solution :

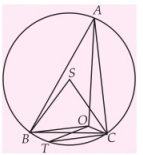


Let α, β, γ be the affixes of A, B, C respectively . Then the affix of the centroid of ΔABC *is* $\frac{1}{3}(\alpha+\beta+\gamma)$.

Now the affixes of X, Y, Z are $\frac{\beta+\gamma}{2}, \frac{\gamma+\alpha}{2}, \frac{\alpha+\beta}{2}$ respectively; therefore the centroid of ΔXYZ is given by $\frac{1}{3}\left(\frac{\beta+\gamma}{2}+\frac{\gamma+\alpha}{2}+\frac{\alpha+\beta}{2}\right)=\frac{1}{3}(\alpha+\beta+\gamma)$; hence both triangle have the same centroid.

5. The diagonals of the quadrilateral $BTCO$ bisect each other at M. Hence $BTCO$ is a parallelogram. Since $CO \perp AB$ and $CO \parallel TB$, we have $TB \perp AB$. Similarly $TC \perp AC$. Thus the circle having AT as diameter passes through B and C. This is the circumcircle of triangle ABC. Let S be the circumcentre. Then $\angle BSC = 2\angle BAC = 60°$ and $\angle SBC = \angle SCB = 60°$; therefore ΔSBC is equilateral. Therefore BC is equal to the radius of the circle. Therefore $AT = 2BC$.

Alternative solution : Let the circumcentre of ΔABC be the origin. Let z_1, z_2, z_3 be complex numbers representing A, B, C respectively. Then the centroid G is $\frac{z_1+z_2+z_3}{3}$.



Since G lies on SO such that $SG : SO = 1 : 3$, O is $z_1+z_2+z_3$. Since M is the mid-point of BC, its coordinate is $\frac{z_2+z_3}{2}$. Since T divides the segment OM in the ratio $2 : -1$, its coordinate is

$$\frac{1}{-1+2}\left(-1(z_1+z_2+z_3)+2\left(\frac{z_2+z_3}{2}\right)\right)=-z_1$$

$\therefore \quad AT = |-2z_1| = 2R$. Also $BC = 2R\sin 30° = R$.

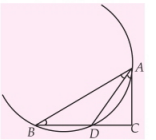
6. Let AD be the internal bisector of $\angle A$, meeting BC at D. Then $\frac{BD}{DC}=\frac{AB}{AC}$; adding 1 to both sides we get

$$\frac{BD}{DC}=\frac{AB+AC}{AC} \qquad \ldots (1)$$

Since $\angle CAD = \angle ABD$, AC is tangent to the circumcircle of ΔABD. Therefore

$$BC \cdot DC = AC^2 \qquad \ldots (2)$$

Now (1) × (2) $\Rightarrow$ $BC^2 = AC(AB + AC)$.



Alternative solution :

$A = 2B \Rightarrow C = \pi - 3B \Rightarrow \sin C = \sin 3B$

Therefore $\sin^2 A = \sin^2 2B = 2\sin B \cos B \sin 2B$

$= \sin B(\sin B + \sin 3B) = \sin B(\sin B + \sin C)$

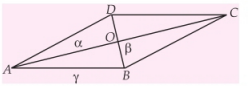
$\Rightarrow \quad a^2 = b(b + c)$

7. Let $\angle OAB$ be θ. Then the area of the rhombus is $\gamma^2 \sin 2\theta$. It is given that $\gamma^2 \sin 2\theta = \frac{1}{2}\gamma^2$. Therefore

$$\sin 2\theta = \frac{1}{2} \Rightarrow 2\theta = 30° \quad \text{or} \quad 150° \Rightarrow \theta = 15° \text{ or } 75°$$

Hence the required ratio is

$$\tan 75° = \tan(45° + 30°) = \frac{(1 + 1/\sqrt{3})}{(1 - 1/\sqrt{3})} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2}$$



Alternative solution : Areas of the square and the rhombus are $\alpha^2 + \beta^2$ and $2\alpha\beta$ respectively. Therefore $\alpha^2 + \beta^2 = 4\alpha\beta \Rightarrow (\alpha + \beta)^2 = 6\alpha\beta$ and $(\alpha - \beta)^2 = 2\alpha\beta$

$$\Rightarrow \left(\frac{\alpha+\beta}{\alpha-\beta}\right)^2 = 3 \Rightarrow \frac{\alpha+\beta}{\alpha-\beta} = \sqrt{3}$$

$$\Rightarrow \frac{\alpha}{\beta} = \frac{\sqrt{3}+1}{\sqrt{3}-1} = \frac{(\sqrt{3}+1)^2}{2} = 2 + \sqrt{3}$$

8. Let α be any real number in (0, 1). Suppose ϕ is a continuous function from R to R such that for all $x \in R$.

$$\phi(x) - 2\phi(\alpha x) + \phi(\alpha^2 x) = 0 \qquad \ldots (1)$$

Then ϕ is a constant function.

Proof (1) can be written as

$$\phi(x) - \phi(\alpha x) = \phi(\alpha x) - \phi(\alpha^2 x) \qquad \ldots (2)$$

For any $n \in N$, replacing x in (1) by αx, $\alpha^2 x$, ..., $\alpha^n x$ we get

$$\phi(x) - \phi(\alpha x) = \phi(\alpha^n x) - \phi(\alpha^{n+1} x) \qquad \ldots (3)$$

Since $\alpha^n \to 0$ as $n \to \infty$, by continuity $\phi(\alpha^n x) \to \phi(0)$

as $n \to \infty$. Therefore (3) yields $\phi(x) - \phi(\alpha x) = 0$ *i.e.*, $\phi(x) = \phi(\alpha x)$. So for any $n \in N$, we have $\phi(x) = \phi(\alpha^n x)$.

Therefore $\phi(x) = \lim_{n\to\infty} \phi(\alpha^n x) = \phi(0)$

i.e., ϕ is a constant function. We are given that

$$f(x) - 2f\left(\frac{x}{2}\right) + f\left(\frac{x}{4}\right) = x^2 \quad \text{... (4)}$$

Choosing $\phi(x) = f(x) - \frac{16}{9}x^2$, we see that (4) transforms into (1) for $\alpha = \frac{1}{2}$. Therefore ϕ is a constant function.

Hence for some $k \in R$, $f(x) = k + \frac{16}{9}x^2$.

9. Let a, b, c be three integers such that

$a^3 + 5b^3 + 25c^3 - 15abc = 0$ ... (1)

We shall show in two ways that

$a = b = c = 0$ ... (2)

Method 1 : The algebraic identity

$$x^3 + y^3 + z^3 - 3xyz = \frac{1}{2}(x+y+z)[(x-y)^2 + (y-z)^2 + (z-x)^2]$$

implies either $a + 5^{1/3}b + 5^{2/3}c = 0$ ... (3)

or $(a - 5^{1/3}b)^2 + (5^{1/3}b - 5^{2/3}c)^2 + (5^{2/3}c - a)^2 = 0$ ... (4)

Since $a, b, c \in Z$, (3) is possible only when (2) holds.

(4) $\Rightarrow$ $a = 5^{1/3}\, b = 5^{2/3}c$; again, this is possible only when (2) holds.

Method 2 : Suppose (a, b, c) is a non-zero solution for the given equation. If a, b, c has any common divisor d, then $(a/d, b/d, c/d)$ is also a solution for the given equation. So we can assume that $gcd\ (a, b, c) = 1$. 5 divides a^3 since it divides all other terms in (1). Therefore $5 | a$; *i.e.*, $a = 5a_1$ for some $a_1 \in z$. Now (1) implies $25a_1^3 + b^3 + 5c^3 - 15a_1bc = 0$

Now by the same reasoning, $5 \mid b$; substituting $b = 5b_1$, we get $5a_1^3 + 25b_1^3 + c^3 - 15a_1b_1c = 0$

which in turn implies that $5 \mid c$. Thus 5 divides all of a, b, c a contradiction of our assumption. Therefore (2) holds.

10. Suppose x, y are two positive integers such that

$$\frac{1}{\sqrt{x}} + \frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}} \quad \text{...(1)}$$

Then $\frac{1}{\sqrt{y}} = \frac{1}{\sqrt{20}} - \frac{1}{\sqrt{x}} = \frac{\sqrt{x} - \sqrt{20}}{\sqrt{20}.\sqrt{x}}$.

Therefore $\frac{1}{y} = \frac{x + 20 - 4\sqrt{5x}}{20x}$

implying that $\sqrt{5x}$ is rational. Now $x \in N \Rightarrow \sqrt{5x} \in N$

Hence $5x$ is the square of an integer which is divisible by 5. Therefore $5x = (5a)^2$ for some $a \in N$. *i.e.*, $x = 5a^2$.

Similarly we can write $y = 5b^2$ for some $b \in N$. Now (1)

$\Rightarrow \frac{1}{a} + \frac{1}{b} = \frac{1}{2} \Rightarrow 2(a+b) = ab$

$\Rightarrow (a-2)(b-2) = 4$

$\Rightarrow (a, b) \in \{(3, 6), (4, 4), (6, 3)\}$

Hence the solution set is {(45, 180), (80, 80), (180, 45)}.

11. The solutions are $(x, y) = (170, 145)$ or $(1827, 145)$.

We have $x^2 + y^2 = 1997(x - y)$

$2(x^2 + y^2) = 2 \times 1997(x - y)$

$x^2 + y^2 + (x^2 + y^2 - 2 \times 1997(x - y)) = 0$

$(x + y)^2 + ((x + y)^2 - 2 \times 1997(x - y)) = 0$

$(x + y)^2 + (1997 - x + y)^2 = 1997^2$.

Since x and y are positive integers, $0 < x + y < 1997$ and $0 < 1997 - x + y < 1997$. Thus the problem reduces to solving $a^2 + b^2 = 1997^2$ in positive integers. Since 1997 is a prime, $gcd(a, b) = 1$. By Pythagorean substitution, there are positive integers $m > n$ such that $gcd(m, n) = 1$ and $1997 = m^2 + n^2$, $a = 2mn$, $b = m^2 - n^2$.

Since m^2, $n^2 \equiv 0, 1, -1 \pmod 5$ and $1997 \equiv 2 \pmod 5$, $m, n \equiv \pm 1 \pmod 5$. Since m^2, $n^2 \equiv 0, 1 \pmod 3$ and $1997 \equiv 2 \pmod 3$, $m, n \equiv +1 \pmod 3$. Therefore $m, n \equiv 1, 4, 11, 14 \pmod{15}$. Since $m > n$, $\frac{1997}{2} \le m^2 \le 1997$.

Thus we only need to consider $m = 34, 41, 44$. The only solution is $(m, n) = (34, 29)$. Thus $(a, b) = (1972, 315)$, which leads to our final solution.

12. We claim that $A = \left\{\begin{vmatrix} x, 1, 1 \\ x, 0 \end{vmatrix}\right\}$ for some real number $x > 1$. We first prove the following lemma.

Lemma : Let A be a set of distinct non-negative real numbers such that

(a) the set A contains at least 4 numbers;

(b) the value of each number in A is less than 1;

(c) for any 4 distinct numbers $a, b, c, d \in A$ the number $ab + cd \in A$.

Proof : Suppose, on the contrary, that A is finite. Let $A = \{a_1, a_2, ..., a_n\}$, where $1 > a_1 > a_2 > ... > a_n \ge 0$ and $n \ge 4$. We first prove that $a_n \ne 0$. Assume for contradiction that $a_n = 0$. Then $a_1a_n + a_{n-1}a_{n-2} \in A$. But $a_1a_n = 0$ and $a_{n-1}a_{n-2} < a_{n-1}$, and hence there is an element of A in between a_{n-1} and a_n, which is impossible. Therefore if A is finite $a_n \ne 0$.

Now, let $a = a_1a_2 + a_3a_4$, $b = a_1a_3 + a_2a_4$, $c = a_1a_4 + a_2a_3$. Then $a, b, c \in A$ and $a > b > c$. If $c > a_4$, then $a = a_1$, $b = a_2$, $c = a_3$. From the first and the third equation, we have

$a_1(1 - a_2) = a_3a_4$

$a_3(1 - a_2) = a_1a_4$

Multiplying the equations gives

$a_1a_3(1 - a_2)^2 = a_3a_1a_4^2$.

Since $a_1, a_3 \neq 0$, we may now divide by a_1a_3 and take square roots to obtain $1 - a_2 = a_4$. But then $a_1 = a_3$, unless $a_4 = 0$. However, a_4 cannot equal zero, and we have reached a contradiction. So $c = a_1a_4 + a_2a_3 \leq a_4$, and $a_4(1 - a_1) \geq a_2a_3$. Thus $a_3 > a_4$ implies that

$a_3(1-a_1) \geq a_2a_3 \Rightarrow 1-a_1 \geq a_2 \Rightarrow 1 \geq a_1 + a_2$

So $a_2 < \frac{1}{2}$ and $a = a_1a_2 + a_3a_4 < \frac{a_1}{2} + \frac{a_3}{2} < a_1$. If $c = a_4$, then $a = a_2$, $b = a_3$, $c = a_4$. From the second equation and the third equation, we have

$a_3(1 - a_1) = a_4a_2$,

$a_4(1 - a_1) = a_2a_3$

Using the same argument as before, we have $1 - a_1 = a_2 = 0$, a contradiction. Thus $c < a_4$ and a_5 exists. If we re-apply the above argument to (a_2, a_3, a_4, a_5), we can prove the existence of a_6 and so on.

Therefore A must be an infinite set.

Now we prove our main result. We use the same notation. From the lemma, we know that in order for A to satisfy the given conditions, we must have $a_1 \geq 1$. Then $c > a_4$ and $a = a_1$, $b = a_2$, $c = a_3$. From the same argument used in the proof of the lemma, we have $1 - a_2 = a_4 = 0 \Leftrightarrow a_2 = 1$ and $a_4 = 0$. Thus $b = a_1a_3 = a_2 = 1$.

We have $A = \left\{\frac{a_1, 1, 1}{a_1, 0}\right\}$, as claimed.

13. We prove that

$$n\left(1+\frac{1}{3}+...+\frac{1}{2n-1}\right) > (n+1)\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2n}\right)$$

by induction. For $n = 2$, $\frac{8}{3} > \frac{9}{4}$. Assume the claim is true for $n = k \geq 2$, *i.e.*, we have

$$k\left(1+\frac{1}{3}+...+\frac{1}{2k-1}\right) > (k+1)\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2k}\right).$$

Now, let $n = k + 1$. Note that

$$\left(1+\frac{1}{3}+...+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$

$$=\left(\frac{1}{2}+\frac{1}{3}+...+\frac{1}{2k-1}\right)+\frac{1}{2}+\frac{k+1}{2k+1}$$

$$>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2k}\right)+\frac{1}{2}+\frac{k+1}{2k+1}$$

$$>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2k}\right)+\frac{k+1}{2k+2}+\frac{1}{2k+1}$$

$$>\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2k}\right)+\frac{k+2}{2k+2}$$

Induction is complete as

$$(k+1)\left(1+\frac{1}{3}+...+\frac{1}{2k-1}+\frac{1}{2k+1}\right)$$

$$=k\left(1+\frac{1}{3}+...+\frac{1}{2k-1}\right)+\left(1+\frac{1}{3}+...+\frac{1}{2k-1}\right)+\frac{k+1}{2k+1}$$

$$>k\left(1+\frac{1}{3}+...+\frac{1}{2k-1}\right)+\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2k}\right)+\frac{k+2}{2k+2}$$

$$>(k+2)\left(\frac{1}{2}+\frac{1}{4}+...+\frac{1}{2k+2}\right)$$

14. The "if" direction of the claim is easily proven by induction on n; we prove the "only if" direction by contradiction. Suppose, on the contrary, that there exist pairs satisfying the equation but not of the described form; let (a, b) be such a pair with minimal sum $a + b$. We claim that $(c, a) = (m^2a - b, a)$ is another such a pair but with smaller sum $c + a$, which leads to a contradiction.

(a) $a = 0$. Then $(a, b) = (0, m) = (a_0, a_1)$, a contradiction.

(b) $a = m$. Then $(a, b) = (m, m^3) = (a_1, a_2)$, a contradiction.

(c) $a = 1$. Then $b \geq a = 1$ and $(b + 1) \mid (b^2 + 1)$; but $(b + 1) \mid (b^2 - 1)$, thus $(b + 1) \mid [(b^2 + 1) - (b^2 - 1)] = 2$.

We have $b = 1$, thus $m = 1$ and $(a, b) = (1, 1) = (a_1, a_2)$, a contradiction.

(d) $2 \leq a < m$. Rewrite $\frac{(a^2+b^2)}{(ab+1)} = m^2$ as

$b^2 - m^2ab + a^2 - m^2 = 0$,

we know that $t = b$ is a root of the quadratic equation

$t^2 - m^2at + a^2 - m^2 = 0$ ... (1)

Thus $m^4a^2 + 4m^2 - 4a^2$ the discriminant of the equation must be a perfect square. But

$(m^2a + 1)^2 = m^4a^2 + 2m^2a + 1$

$> m^4a^2 + 4m^2 - 4a^2 > (m^4a)^2$

For $2 \leq a < m$. So the discriminant cannot be a perfect square, a contradiction.

(e) $a > m$. Again $t = b$ is a root of (1). It is easy to check that $t = m^2a - b = c$ also satisfies the equation. We have $bc = a^2 - m^2 > 0$; since $b \geq 0$, $c > 0$. Since $a > 0$ and $c > 0$, $ac + 1 > 0$, we have $\frac{c^2+a^2}{ca+1} = m^2$.

Since $c > 0$, $b \geq a$ and $bc = a^2 - m^2 < a^2$, we have $c < a$. Thus (c, a) is a valid pair. Also, it cannot be of the form (a_n, a_{n+1}) or else

$(a, b) = (a_{n+1}, m^2a_{n+1} - a_n) = (a_{n+1}, a_{n+2})$.

But then, $c + a < a + a \leq b + a$, as desired.

From the above, we see that our assumption is false. Therefore every pair satisfying the original equation must be of the described form.

15. The solutions are $n = 3, 7, 23$. Since 2 is a prime, $1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3} = 2^k$ for some positive integer $k \leq 2000$.

We have $1+\binom{n}{1}+\binom{n}{2}+\binom{n}{3}=(n+1)\frac{(n^2-n+6)}{6}$,

i.e., $(n+1)(n^2-n+6)=3\times 2^{k+1}$. Let $m=n+1$, then $m\geq 4$ and $m(m^2-3m+8)=3\times 2^{k+1}$. We consider the following two cases.

(a) $m=2^s$. Since $m\geq 4$, $s\geq 2$, we have

$2^{2s}-3\times 2^s+8=m^2-3m+8=3\times 2^t$

for some positive integer t.

If $s\geq 4$, then $8\equiv 3\times 2^t \pmod{16} \Rightarrow 2^t=8$

$\Rightarrow m^2-3m+8=24\Rightarrow m(m-3)=16$, which is impossible. Thus either

$s=3$, $m=8$, $t=4$, $n=7$, or $s=2$, $m=4$, $t=2$, $n=3$.

(b) $m=3\times 2^n$. Since $m\geq 4$, $m>4$ and $u\geq 1$. We have $9\times 2^{2u}-9\times 2^u+8=m^2+8=2^v$ for some positive integer v. It is easy to check that there is no solution for v when $u=1,2$. If $u\geq 4$, we have $8\equiv 2^v \pmod{16}$ $\Rightarrow v=3$ and $m(m-3)=0$, which is impossible. So $u=3$, $m=3\times 2^3=24$, $v=9$, $n=23$.

16. Let $f(x)=x\lfloor x\lfloor x\rfloor\rfloor$

Lemma: Let a and b be real numbers. If a and b have the same sign and $|a|>|b|\geq 1$, then $|f(a)|>|f(b)|$.

Proof: We notice that $\lfloor a\rfloor\geq\lfloor b\rfloor\geq 1$. Multiplying this by $|a|>|b|\geq 1$, we have $|a\lfloor a\rfloor|>|b\lfloor b\rfloor|\geq 1$. Notice that $a\lfloor a\rfloor$ and $a\lfloor a\lfloor a\rfloor\rfloor$ have the same signs as $b\lfloor b\rfloor$ and $b\lfloor b\lfloor b\rfloor\rfloor$ respectively. In a similar manner, $|a\lfloor a\lfloor a\rfloor\rfloor|>|b\lfloor b\lfloor b\rfloor\rfloor|\geq 1$, $|a\lfloor a\lfloor a\lfloor a\rfloor\rfloor\rfloor|\geq|b\lfloor b\lfloor b\lfloor b\rfloor\rfloor\rfloor|\geq 1$, and $|f(a)|>|f(b)|$, as claimed.

We have $f(x)=0$ for $|x|<1$, $f(1)=f(-1)=1$. Suppose that $f(x)=88$. So $|x|>1$, and we consider the following two cases.

(a) $x\geq 1$. It is easy to check that $f\left(\frac{22}{7}\right)=88$. From the lemma, we know that $f(x)$ is increasing for $x>1$. So $x=\frac{22}{7}$ is the unique solution on this interval.

(b) $x\leq -1$. From the lemma, $f(x)$ is decreasing for $x<1$. Since $|f(-3)|=81<f(x)=88<\left|f\left(-\frac{112}{37}\right)\right|=112$, $-3>x>-\frac{112}{37}$ and $\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor=-37$.

But then $x=-\frac{88}{37}>-3$ a contradiction. So there is no solution on this interval.

Therefore, $x=\frac{22}{7}$ is the only solution.

Note : $\frac{22}{7}$ and $-\frac{112}{37}$ are found by finding $\lfloor x\rfloor$, $\lfloor x\lfloor x\rfloor\rfloor$, and $\lfloor x\lfloor x\lfloor x\rfloor\rfloor\rfloor$ in that order. For example, for $x\geq 1$, $f(3)<88<f(4)$ so $3<x<4$. Then $\lfloor x\rfloor=3$ and $x\lfloor x\lfloor 3x\rfloor\rfloor=88$.

Then $f(3)<88<f\left(\frac{10}{3}\right)$ so $\lfloor x\lfloor x\rfloor\rfloor=9$, and so on.

17. 1st solution: Let *STUV* be the trapezoid with *ST* > *UV*, *ST* || *UV*; let *B* and *C* be the midpoints of *ST* and *UV*, respectively; let the diagonals *TV* and *SU* meet at *P*; and let *O* be the centre of *k*. Also, let *l* be the line through *P* parallel to *ST* and *UV*, meeting *TU* and *SV* at *E* and *F*, respectively.

Now, *TU* and *SV* meet at *A*, and *STUV* must be isosceles. From parallel lines,

$$\frac{UE}{ET}=\frac{CP}{PB} \quad \text{and} \quad \frac{UA}{AT}=\frac{CA}{AB}=\frac{UC}{TB}$$

But from similar triangles *CUP* and *BTP* (again by parallel lines), these fractions are equal. So, (*UTEA*) is a harmonic range. Similarly, (*VSFA*) is a harmonic range. In other words, line *EF* is the pole of *A* under the polar map with respect to *k*.

Therefore, *P* is the intersection of this pole and *AC*, independent of the choice of trapezoid.

2nd solution: Define *STUV* and all other points as in the first solution.

Let $AV=v$, $AS=s$, $AO=o$, and r be the radius of k;

Let $\frac{VC}{SB}=x$; $\angle SAC=\alpha$.

From parallel lines, $v=xs$, $VS=(1-x)s$,

$$BP=\frac{BC}{1+x}=\frac{VS\cos\alpha}{1+x}=\frac{(1-x)s\cos\alpha}{1+x}$$

So,

$$AP=AB-BP=s\cos\alpha-\frac{(1-x)s\cos\alpha}{1+x}=\frac{2x}{1+x}s\cos\alpha.$$

From power of a point, $sv=o^2-r^2$. Thus $x=\frac{(o^2-r^2)}{s^2}$.

Applying the law of cosines to triangle *AOS*, we have $\cos\alpha=\frac{s^2+o^2-r^2}{2so}$, which implies that $AP=\frac{(o^2-r^2)}{o}$, independent of the choice of *STUV*.

18. With operations all taken modulo 2, we have

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \Rightarrow \begin{bmatrix} b+d & a+e+c & b+f \\ a+e+g & b+d+h+f & c+e+i \\ d+h & g+e+i & h+f \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} c+g & b+h & a+i \\ d+f & 0 & d+f \\ a+i & b+h & c+g \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} b+d+h+f & c+g+a+i & b+d+h+f \\ c+g+a+i & 0 & c+g+a+i \\ b+d+h+f & a+c+g+i & b+d+h+f \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Thus regardless of the starting position, we always obtain a table with all entries even in at most 4 steps. On the other hand,

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

has all entries even after one operation; further operations keep all the entries even.

Note that $b + d$, $h + f$, $b + d + h + f$ are the main diagonal elements after the first operation. Of these three elements, at least one is even. So, we never obtain a table with all entries odd.

19. Triangles ABC and $A'B'C'$ are similar. Let $k = \frac{AB}{A'B'} = \frac{BC}{B'C'} = \frac{CA}{C'A'}$. Thus $[ABC] = k^2[A'B'C']$. Let d be the distance between the parallel lines AB and $A'B'$. In trapezoid $AA'B'B$, triangles $AB'B$ and $AA'B'$ have the same height d corresponding to the respective bases AB and $A'B'$; triangles $PA'B'$ have the same height d corresponding to the common base $A'B'$. We have

$[AB'B] = k[AA'B'] = k[PA'B']$, $[AA'B'B] = (k + 1)[PA'B']$

Similarly, we have

$[BB'C'C] = (k + 1)[QB'C']$ and $[CC'A'A] = (k + 1)[RC'A']$

Thus

$(k + 1)(k - 1)[A'B'C'] = (k^2 - 1)[A'B'C']$

$= [ABC] - [A'B'C']$

$= [AA'B'B] + [BB'C'C] + [CC'A'A]$

$= (k + 1)([PA'B'] + [QB'C'] + [RC'A'])$

$= (k + 1)([PQR] - [A'B'C'])$

So $(k - 1)[A'B'C'] = [PQR] - [A'B'C']$ and

$$\frac{[PQR]}{[A'B'C']} = k = \frac{AB}{A'B'}, \text{ as desired.}$$

20. (a) $(a^2 + 3b^2)(c^2 + 3d^2) = (ac + 3bd)^2 + 3(ad - bc)^2$

(b) $7 = (\pm 2)^2 + 3(1)^2$, so reversing the above construction may yield the desired result. Indeed, if $7n = e^2 + 3f^2$, then solving for c and d after setting $a = 2$, $b = 1$ and $a = -2$, $b = 1$ gives

$$c_1 = \frac{2e - 3f}{7}, d_1 = \frac{e + 2f}{7}; \; c_2 = \frac{2e + 3f}{7}, d_2 = \frac{e - 2f}{7}.$$

With all congruences taken modulo 7, we have

$$e^2 + 3f^2 \equiv 0 \Leftrightarrow e^2 \equiv 4f^2 \Leftrightarrow e \equiv \pm 2f.$$

If $e \equiv -2f$, then c_1 and d_1 are integers; if $e \equiv 2f$, then c_2 and d_2 are integers. Either way, n can be written in the desired form.

21. The largest p is 5. Note that b is even, so we may write $b = 2c$.

Now $p = \frac{c}{2}\sqrt{\frac{a-c}{a+c}}$ or $\frac{4p^2}{c^2} = \frac{a-c}{a+c}$.

Write $\frac{2p}{c} = \frac{m}{n}$ in lowest terms. If $k = gcd(a - c, a + c)$, then $a - c = km^2$, $a + c = kn^2$, $a + c = kn^2$,

so $2c = k(n^2 - m^2)$ and $4pn = km(n^2 - m^2)$.

In case m, n are both odd, then $8 \mid m^2 - n^2$ and so p is even, that is, $p = 2$. On the other hand, if m and n are not both odd, $n^2 - m^2$ is odd, so k must be even. Write $k = 2r$, so that we have $2pn = rm(n^2 - m^2)$. Now n is co-prime to m and to $n^2 - m^2$, so n divides r. Writing $r = ns$, we have $2p = s(n - m)(n + m)m$.

Suppose $p > 2$. Since $m + n$ and $n - m$ are odd, $m + n = p$ and $n - m = 1$. Thus s, m are each at most 2. This leaves only the possibilities $(m, n) = (1, 2)$ or $(2, 3)$. In either case, $p \le 5$, and fortunately this can be achieved with $m = 2$, $n = 3$, $s = 1$, $r = 3$, $k = 6$, $c = 15$, $b = 30$, and $a = 39$.

22. By Cauchy-Schwartz inequality,

$$\sqrt{x+y+z}\sqrt{\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z}} \ge \sqrt{x-1}+\sqrt{y-1}+\sqrt{z-1}$$

On the other hand, by hypothesis,

$$\frac{x-1}{x}+\frac{y-1}{y}+\frac{z-1}{z} = 1, \text{ so the desired result follows.}$$

23. We proceed inductively on n.

For $n = 1$, there is 1 permutation that can be reached by 0 transpositions and both sides of the equation is equal to 1.

Assume that the equation is true for $n = k$. For $n = k + 1$, let $p_{k+1,\,m}$ be a permutation of $\{1, ..., k + 1\}$ which can be written as the product of m transpositions but not of $m - 1$ transpositions and $k + 1$ is in the $(k + 1 - h)^{\text{th}}$ position. Then it takes at least h transpositions to get $k + 1$ to the $(k + 1 - h)^{\text{th}}$ position. Therefore $p_{k+1,\,m} - k + 1$ is a permutation of $\{1, ..., k\}$ which can be written as the product of $m - h$ transpositions but not of $m - h - 1$ transpositions. Clearly, this correspondence is a bijection. So we have $c_{k+1,\,m} = c_{k,m} + c_{k,\,m-1} + ... + c_{k,\,m-k}$. Therefore

$$\prod_{i=1}^{k+1}(1 + t + \cdots + t^{i-1}) = \left(\sum_{m=0}^{\infty} c_{k,m}t^m\right)(1 + t + \cdots + t^k)$$

$$= \sum_{m=0}^{\infty}(c_{k,m} + c_{k,m-1} + \cdots + c_{k,m-k})t^m = \sum_{m=0}^{\infty} c_{k+1,m}t^m$$

and the induction is complete. ■■

SOLVED PAPER
CBSE CLASS XII
BOARD EXAMINATION 2014

GENERAL INSTRUCTIONS

(i) All questions are compulsory.
(ii) The question paper consists of 29 questions divided into three sections A, B and C. Section-A comprises of 10 questions of one mark each, Section-B comprises of 12 questions of four marks each and Section-C comprises of 7 questions of six marks each.
(iii) All questions in Section-A are to be answered in one word, one sentence or as per the exact requirement of the question.
(iv) There is no overall choice. However, internal choice has been provided in 4 questions of four marks each and 2 questions of six marks each. You have to attempt only one of the alternatives in all such questions.
(v) Use of calculators is not permitted. You may ask for logarithmic tables, if required.

Time : 3 hrs. Marks : 100

SECTION-A

1. If $2\begin{bmatrix}3 & 4\\ 5 & x\end{bmatrix}+\begin{bmatrix}1 & y\\ 0 & 1\end{bmatrix}=\begin{bmatrix}7 & 0\\ 10 & 5\end{bmatrix}$, find $(x-y)$.

[From MTG Excel in Mathematics, Similar Question, Page 77, Q. 9]

2. Solve the following matrix equation for x:

$$[x\ \ 1]\begin{bmatrix}1 & 0\\ -2 & 0\end{bmatrix}=O$$

3. If $\begin{vmatrix}2x & 5\\ 8 & x\end{vmatrix}=\begin{vmatrix}6 & -2\\ 7 & 3\end{vmatrix}$, write the value of x.

[From MTG Excel in Mathematics, Similar Question, Page 120, Q. 2]

4. Write the anti-derivative of $\left(3\sqrt{x}+\frac{1}{\sqrt{x}}\right)$

[From MTG Excel in Mathematics, Similar Question, Page 372, Q. 21]

5. If $\sin\left(\sin^{-1}\frac{1}{5}+\cos^{-1}x\right)=1$, then find the value of x.

[From Excel in Mathematics, Page 47, Q. 14]

6. Let * be a binary operation, on the set of all non-zero real numbers, given by $a * b = \frac{ab}{5}$ for all $a, b \in R - \{0\}$. Find the value of x, given that $2 * (x * 5) = 10$.

7. Find the projection of vector $\hat{i}+3\hat{j}+7\hat{k}$ on the vector $2\hat{i}-3\hat{j}+6\hat{k}$.

8. Write the vector equation of the plane, passing through the point (a, b, c) and parallel to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k})=2$

9. Evaluate: $\int_0^{\pi/2}e^x(\sin x-\cos x)dx$

10. Write a unit vector in the direction of the sum of the vectors $\vec{a}=2\hat{i}+2\hat{j}-5\hat{k}$ and $\vec{b}=2\hat{i}+\hat{j}-7\hat{k}$.

SECTION-B

11. Prove that, for any three vectors $\vec{a}$, $\vec{b}$, $\vec{c}$

$$[\vec{a}+\vec{b},\ \vec{b}+\vec{c},\ \vec{c}+\vec{a}]=2[\vec{a},\ \vec{b},\ \vec{c}]$$

OR

Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ are such that $\vec{a}+\vec{b}+\vec{c}=\vec{0}$ and $|\vec{a}|=3$, $|\vec{b}|=5$ and $|\vec{c}|=7$. Find the angle between $\vec{a}$ and $\vec{b}$.

12. Solve the following differential equation:

$$(x^2-1)\frac{dy}{dx}+2xy=\frac{2}{x^2-1}$$

13. Evaluate: $\int\frac{\sin^6x+\cos^6x}{\sin^2x\cdot\cos^2x}dx$

[From MTG Excel in Mathematics, Page 361, Q. 1(ii)]

OR

Evaluate : $\int (x-3)\sqrt{x^2+3x-18}\,dx$

14. Find the intervals in which the function $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$ is

(a) strictly increasing (b) strictly decreasing

[From MTG Excel in Mathematics, Similar Questions, Page 297, Q. 11]

OR

Find the equation of the tangent and normal to the curve $x = a\sin^3\theta$ and $y = a\cos^3\theta$ at $\theta = \frac{\pi}{4}$.

[From MTG Excel in Mathematics, Similar Question, Page 313, Q. 70]

15. Let $A = \{1, 2, 3, ..., 9\}$ and R be the relation in $A \times A$ defined by $(a, b)\ R\ (c, d)$ if $a + d = b + c$ for $(a, b), (c, d)$ in $A \times A$. Prove that R is an equivalence relation. Also obtain the equivalence class $[(2, 5)]$.

[From MTG Excel in Mathematics, Short Answer Type Questions, Page 31, Q. 3]

16. Prove that

$$\cot^{-1}\left(\frac{\sqrt{1+\sin x}+\sqrt{1-\sin x}}{\sqrt{1+\sin x}-\sqrt{1-\sin x}}\right) = \frac{x}{2};\ x \in \left(0, \frac{\pi}{4}\right).$$

[From MTG Excel in Mathematics, Long Answer Type Questions, Page 52, Q. 12]

OR

Prove that $2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right) = \frac{\pi}{4}$

17. If $y = x^x$, prove that $\frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$

18. Assume that each born child is equally likely to be a boy or a girl. If a family has two children, what is the conditional probability that both are girls? Given that

(i) the youngest is a girl. (ii) atleast one is a girl.

[From MTG Excel in Mathematics, Page 725, Q. 12]

19. Using properties of determinants, prove the following:

$$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = 1 + x^2 + y^2 + z^2$$

[From MTG Excel in Mathematics, Similar Question, Page 159, Q. 11]

20. Differentiate $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\sin^{-1}\left(\frac{2x}{1+x^2}\right)$, when $x \neq 0$.

21. Find the particular solution of the differential equation $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$, given that $y = \frac{\pi}{2}$, when $x = 1$.

22. Show that lines $\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$ intersects. Also, find their point of intersection.

[From MTG Excel in Mathematics, Page 611, Q. 11]

23. A dealer in rural area wishes to purchase a number of sewing machines. He has only ₹ 5760 to invest and has space for at most 20 items for storage. An electronic sewing machine cost him ₹ 360 and a manually operated sewing machine ₹ 240. He can sell an electronic sewing machine at a profit of ₹ 22 and a manually operated sewing machine at a profit of ₹ 18. Assuming that he can sell all the items that he can buy, how should he invest his money in order to maximise his profit? Make it as a LPP and solve graphically.

[From MTG Excel in Mathematics, Similar Question, Page 669, Q. 12]

24. A card from a pack of 52 playing cards is lost. From the remaining cards of the pack three cards are drawn at random (without replacement) and are found to be all spades. Find the probability of the lost card being a spade.

[From MTG Excel in Mathematics, Similar Question, Page 757, Q. 47]

OR

From a lot of 15 bulbs which include 5 defectives, a sample of 4 bulbs is drawn one by one with replacement. Find the probability distribution of number of defective bulbs. Hence, find the mean of the distribution.

[From MTG Excel in Mathematics, Similar Questions, Page 733, Q. 6]

25. Find the area of the region in the 1st quadrant enclosed by the x-axis, the line $y = x$ and the circle $x^2 + y^2 = 32$.

[From MTG Excel in Mathematics, Page 482, Q. 8]

26. Find the distance between the point (7, 2, 4) and the plane determined by the points $A(2, 5, -3)$, $B(-2, -3, 5)$ and $C(5, 3, -3)$.

[From MTG Excel in Mathematics, Similar Question, Page 642, Q. 22]

OR

Find the distance of the point (–1, –5, –10) from the point of intersection of the line $\vec{r} = 2\hat{i} - \hat{j} + 2\hat{k} + \lambda(3\hat{i} + 4\hat{j} + 2\hat{k})$ and the plane $\vec{r} \cdot (\hat{i} - \hat{j} + \hat{k}) = 5$.

[From MTG Excel in Mathematics, Page 628, Q. 18]

27. Two schools P and Q want to award their selected students on the values of Discipline, Politeness and Punctuality. The school P wants to award ₹ x each, ₹ y each and ₹ z each for the three respective values to its 3, 2 and 1 students with a total award money of ₹ 1,000. School Q wants to spend ₹ 1,500 to award its 4, 1 and 3 students on the respective values (by giving the same award money for the three values as before). If the total amount of

awards for one prize on each value is ₹ 600, using matrices, find the award money for each value. Apart from the above three values, suggest one more value for awards.

28. Evaluate: $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx$

[From MTG Excel in Mathematics, Page 428, Q. 134]

29. Of all the closed right circular cylindrical cans of volume 128π cm^3, find the dimensions of the can which has minimum surface area.

[From MTG Excel in Mathematics, Similar Question, Page 287, Q. 21]

SOLUTIONS

1. We have, $2\begin{bmatrix} 3 & 4 \\ 5 & x \end{bmatrix} + \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

On solving, we get

$\Rightarrow \begin{bmatrix} 7 & 8+y \\ 10 & 2x+1 \end{bmatrix} = \begin{bmatrix} 7 & 0 \\ 10 & 5 \end{bmatrix}$

On comparing corresponding elements of the above matrices, we get

$8 + y = 0$ and $2x + 1 = 5$

$\Rightarrow y = -8 \quad \Rightarrow \quad x = 2$

$\therefore x - y = 2 + 8 = 10$

2. Given, $[x \ 1]\begin{bmatrix} 1 & 0 \\ -2 & 0 \end{bmatrix} = O$

$\Rightarrow [x - 2 \quad 0] = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2$

3. Given, $\begin{vmatrix} 2x & 5 \\ 8 & x \end{vmatrix} = \begin{vmatrix} 6 & -2 \\ 7 & 3 \end{vmatrix}$

$\Rightarrow 2x^2 - 40 = 18 + 14$

$\Rightarrow 2x^2 = 72 \Rightarrow x^2 = 36 \Rightarrow x = \pm 6$

4. $\int \left(3\sqrt{x} + \frac{1}{\sqrt{x}}\right) dx = 3\int x^{1/2} dx + \int x^{-1/2} dx$

$= 2x^{\frac{3}{2}} + 2x^{\frac{1}{2}} + C$

$= 2x^{1/2}(x + 1) + C$

5. We have, $\sin\left(\sin^{-1}\frac{1}{5} + \cos^{-1} x\right) = 1,$

$\Rightarrow \sin^{-1}\frac{1}{5} + \cos^{-1} x = \sin^{-1}(1) = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{1}{5} + \frac{\pi}{2} - \sin^{-1} x = \frac{\pi}{2}$

$\Rightarrow \sin^{-1}\frac{1}{5} = \sin^{-1} x \quad \Rightarrow \quad x = \frac{1}{5}$

6. We have, $2 * (x * 5) = 10$

$\Rightarrow 2 * \left(\frac{5x}{5}\right) = 10 \Rightarrow 2 * x = 10 \Rightarrow \frac{2x}{5} = 10$

$\Rightarrow 2x = 50 \Rightarrow x = 25$

7. Let $\vec{a} = \hat{i} + 3\hat{j} + 7\hat{k}$ and $\vec{b} = 2\hat{i} - 3\hat{j} + 6\hat{k}$

$\therefore$ Required projection

$= \frac{\vec{a} \cdot \vec{b}}{|\vec{b}|} = \frac{1 \times 2 + 3 \times (-3) + 7 \times 6}{\sqrt{2^2 + (-3)^2 + 6^2}} = \frac{35}{\sqrt{49}} = \frac{35}{7} = 5$

8. The direction ratios of normal to the given plane

$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = 2$ are $< 1, 1, 1 >$

So cartesian equation of plane passing through (a, b, c) and having d.r.'s $<1, 1, 1>$ is

$\Rightarrow 1 \cdot (x - a) + 1 \cdot (y - b) + 1 \cdot (z - c) = 0$

$\Rightarrow x + y + z = a + b + c$

$\therefore$ Vector equation of the required plane is,

$\vec{r} \cdot (\hat{i} + \hat{j} + \hat{k}) = a + b + c$

9. Let $I = \int_0^{\pi/2} e^x (\sin x - \cos x) dx$

$\left[\because \int_0^a f(a) dx = \int_0^a f(a - x) dx\right]$

$= \int_0^{\pi/2} e^x \{\cos x + (-\sin x)\} dx$

Put $\cos x = t \Rightarrow -\sin x \, dx = dt$

$I = [e^x \cos x \, dx]_0^{\pi/2}$

$\left[\because \int e^x [f(x) + f'(x)] dx = e^x f(x) + c\right]$

$= (e^{\pi/2} \times 0) - (e^0 \times 1) = -1$

10. Now, $\vec{a} + \vec{b} = 4\hat{i} + 3\hat{j} - 12\hat{k}$

$\therefore \quad |\vec{a} + \vec{b}| = \sqrt{4^2 + 3^2 + (-12)^2} = \sqrt{169} = 13$

$\therefore$ Unit vector, $\hat{c} = \frac{\vec{a} + \vec{b}}{|\vec{a} + \vec{b}|} = \frac{4}{13}\hat{i} + \frac{3}{13}\hat{j} - \frac{12}{13}\hat{k}$

11. We have $[\vec{a} + \vec{b}, \ \vec{b} + \vec{c}, \ \vec{c} + \vec{a}]$

$= (\vec{a} + \vec{b}) \cdot \{(\vec{b} + \vec{c}) \times (\vec{c} + \vec{a})\}$

$= (\vec{a} + \vec{b}) \cdot \{(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{c}) + (\vec{c} \times \vec{a})\}$

$= (\vec{a} + \vec{b}) \cdot \{(\vec{b} \times \vec{c}) + (\vec{b} \times \vec{a}) + (\vec{c} \times \vec{a})\} \quad [\because \vec{c} \times \vec{c} = 0]$

$= \vec{a} \cdot (\vec{b} \times \vec{c}) + \vec{a} \cdot (\vec{b} \times \vec{a}) + \vec{a} \cdot (\vec{c} \times \vec{a}) +$

$\vec{b} \cdot (\vec{b} \times \vec{c}) + \vec{b} \cdot (\vec{b} \times \vec{a}) + \vec{b} \cdot (\vec{c} \times \vec{a})$

$= [\vec{a}, \ \vec{b}, \ \vec{c}] + [\vec{b}, \ \vec{c}, \ \vec{a}]$ [$\because$ Scalar triple product with two equal vector is 0]

$= [\vec{a}, \ \vec{b}, \ \vec{c}] + [\vec{a}, \ \vec{b}, \ \vec{c}]$ [$\because [\vec{b}, \ \vec{c}, \ \vec{a}] = [\vec{a}, \ \vec{b}, \ \vec{c}]$]

$= 2[a \ b \ c]$

OR

Given $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ and $|\vec{a}| = 3, \ |\vec{b}| = 5, \ |\vec{c}| = 7$

We have $\vec{a} + \vec{b} + \vec{c} = \vec{0}$

$\Rightarrow \vec{a} + \vec{b} = -\vec{c} \ \Rightarrow |\vec{a} + \vec{b}|^2 = |-\vec{c}|^2$

$\Rightarrow |\vec{a}|^2 + |\vec{b}|^2 + 2(\vec{a} \cdot \vec{b}) = |\vec{c}|^2$

$\Rightarrow 9+25+2|\vec{a}||\vec{b}|\cos\theta = 49$

$\Rightarrow 2\times 3\times 5\times\cos\theta = 49-34$

$\Rightarrow \cos\theta = \frac{15}{30} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$ or $\frac{5\pi}{3}$

12. $(x^2-1)\frac{dy}{dx}+2xy = \frac{2}{x^2-1}$

$\Rightarrow \frac{dy}{dx}+\frac{2xy}{x^2-1} = \frac{2}{(x^2-1)^2}$

This is a linear differential equation of the form,

$\frac{dy}{dx}+Py = Q$, where $P = \frac{2x}{x^2-1}$ and $Q = \frac{2}{(x^2-1)^2}$

$\therefore$ I.F. $= e^{\int P dx} = e^{\int \frac{2x}{x^2-1}dx} = e^{\log(x^2-1)} = x^2-1$

Hence, differential equation is given by

$y(x^2-1) = \int \frac{2(x^2-1)}{(x^2-1)^2}dx + C$

$\Rightarrow y(x^2-1) = 2\int \frac{dx}{x^2-1} + C$

$\Rightarrow y(x^2-1) = 2\times\frac{1}{2}\log\left|\frac{x-1}{x+1}\right| + C$

$\Rightarrow y(x^2-1) = \log\left|\frac{x-1}{x+1}\right| + C$

13. $\int \frac{\sin^6 x+\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$

$= \int \frac{\sin^6 x}{\sin^2 x\cdot\cos^2 x}dx + \int \frac{\cos^6 x}{\sin^2 x\cdot\cos^2 x}dx$

$= \int \frac{\sin^4 x}{\cos^2 x}dx + \int \frac{\cos^4 x}{\sin^2 x}dx$

$= \int \frac{\sin^2 x(1-\cos^2 x)}{\cos^2 x}dx + \int \frac{\cos^2 x(1-\sin^2 x)}{\sin^2 x}dx$

$= \int\tan^2 x\,dx - \int\sin^2 x\,dx + \int\cot^2 x\,dx - \int\cos^2 x\,dx$

$= \int(\sec^2 x-1)dx + \int(\operatorname{cosec}^2 x-1)\,dx$

$\quad - \int\sin^2 x\,dx - \int(1-\sin^2 x)\,dx$

$= \tan x - x + (-\cot x) - x - x + C$

$= \tan x - \cot x - 3x + C$

OR

$I = \int(x-3)\sqrt{x^2+3x-18}\,dx$

$= \int x\sqrt{x^2+3x-18}\,dx - 3\int\sqrt{x^2+3x-18}\,dx$

$= \int\frac{1}{2}(2x+3-3)\sqrt{x^2+3x-18}\,dx - 3\int\sqrt{x^2+3x-18}\,dx$

$= \frac{1}{2}\int(2x+3)\sqrt{x^2+3x-18}\,dx - \frac{3}{2}\int\sqrt{x^2+3x-18}\,dx$

$\quad -3\int\sqrt{x^2+3x-18}\,dx$

$= \frac{1}{2}\int\underbrace{(2x+3)\times\sqrt{x^2+3x-18}}_{I_1}\,dx - \frac{9}{2}\int\underbrace{\sqrt{x^2+3x-18}}_{I_2}dx$

$= I_1 - \frac{9}{2}I_2$

For I_1, put $x^2+3x-18 = z \Rightarrow (2x+3)dx = dz$

$\therefore\ I_1 = \frac{1}{2}\int\sqrt{z}\,dz = \frac{1}{2}\frac{z^{(1/2)+1}}{\frac{1}{2}+1} + C_1 = \frac{1}{2}\times\frac{2}{3}z^{3/2} + C_1$

$= \frac{1}{3}(x^2+3x-18)^{\frac{3}{2}} + C_1$

For $I_2 = \int\sqrt{x^2+3x-18}\,dx$

$= \int\sqrt{x^2+3x+\left(\frac{3}{2}\right)^2-\left(\frac{3}{2}\right)^2-18}\,dx$

$= \int\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{81}{4}}\,dx = \int\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}\,dx$

$= \frac{x+\frac{3}{2}}{2}\sqrt{\left(x+\frac{3}{2}\right)^2-\left(\frac{9}{2}\right)^2}$

$\quad -\frac{81/4}{2}\log\left|\left(x+\frac{3}{2}\right)+\sqrt{\left(x+\frac{3}{2}\right)^2-\frac{81}{4}}\right| + C_2$

$= \frac{2x+3}{4}\sqrt{x^2+3x-18} - \frac{81}{8}\log\left|\left(x+\frac{3}{2}\right)\right.$

$\quad \left. +\sqrt{x^2+3x-18}\right| + C_2$

The given integral is written as

$I = \frac{1}{3}(x^2+3x-18)^{3/2} - \frac{9}{8}(2x+3)\sqrt{x^2+3x-18}$

$\quad + \frac{729}{16}\log\left|\frac{(2x+3)}{2} + \sqrt{x^2+3x-18}\right| + C$

where $C = C_1 + C_2$ is a constant

14. We have, $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$

$f'(x) = 12x^3 - 12x^2 - 24x = 12x(x^2 - x - 2)$

$\Rightarrow\quad f'(x) = 12x(x+1)(x-2)$

Now, $f'(x) = 0$

$\Rightarrow\quad 12x(x+1)(x-2) = 0 \Rightarrow x = -1,\ x = 0$ or $x = 2$

Hence these points divide the whole real line into four disjoint open intervals namely $(-\infty, -1)$, $(-1, 0)$, $(0, 2)$ and $(2, \infty)$

Interval	Sign of $f'(x)$	Nature of function
$(-\infty, -1)$	$(-)(-)(-) = -$	Strictly decreasing
$(-1, 0)$	$(-)(+)(-) = +$	Strictly increasing
$(0, 2)$	$(+)(+)(-) = -$	Strictly decreasing
$(2, \infty)$	$(+)(+)(+) = +$	Strictly increasing

(a) $f(x)$ is strictly increasing on $(-1, 0) \cup (2, \infty)$

(b) $f(x)$ is strictly decreasing on $(-\infty, -1) \cup (0, 2)$

OR

We have, $x = a\sin^3\theta$; $y = a\cos^3\theta$

$\frac{dx}{d\theta} = 3a\sin^2\theta\cos\theta$ and $\frac{dy}{d\theta} = 3a\cos^2\theta(-\sin\theta)$

$\therefore \frac{dy}{dx} = \frac{-3a\sin\theta\cos^2\theta}{3a\sin^2\theta\cos\theta} = -\frac{\cos\theta}{\sin\theta} = -\cot\theta$

$\therefore \left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{4}} = -\cot\frac{\pi}{4} = -1$

Hence, the equation of the tangent at $\theta = \frac{\pi}{4}$ is

$y - a\cos^3\theta = (-1)(x - a\sin^3\theta)$

$\Rightarrow x + y = a\left\{\left(\frac{1}{\sqrt{2}}\right)^3 + \left(\frac{1}{\sqrt{2}}\right)^3\right\} = \frac{a}{\sqrt{2}}$

$\Rightarrow \sqrt{2}x + \sqrt{2}y = a$

and the equation of the normal is

$y - a\cos^3\theta = \frac{1}{-(-1)}(x - a\sin^3\theta)$

$\therefore y - a\cos^3\theta = (x - a\sin^3\theta)$

$\Rightarrow x - y = a\left(\left(\frac{1}{\sqrt{2}}\right)^3 - \left(\frac{1}{\sqrt{2}}\right)^3\right)$

$\Rightarrow x - y = 0$

15. Given $A = \{1, 2, 3, 4, ..., 9\} \subset N$, the set of natural numbers.

To show that R is an equivalence relation

Reflexivity:

Let (a, b) be an arbitrary element of $A \times A$. Then, we have: $(a, b) \in A \times A$

$\Rightarrow a, b \in A$

$\Rightarrow a + b = b + a$ (by commutativity of addition on $A \subset N$)

$\Rightarrow (a, b)\ R\ (a, b)$

Thus, $(a, b)\ R\ (a, b)$ for all $(a, b) \in A \times A$

So, R is reflexive.

Symmetry:

Let $(a, b), (c, d) \in A \times A$ such that $(a, b)\ R\ (c, d)$.

$a + d = b + c$

$\Rightarrow b + c = a + d$

$\Rightarrow c + b = d + a$ (by commutativity of addition on $A \subset N$)

$\Rightarrow (c, d)\ R\ (a, b)$

Thus, $(a, b)\ R\ (c, d) \Rightarrow (c, d)\ R\ (a, b)$ for all $(a, b), (c, d) \in A \times A$. So, R is reflexive.

Transitivity:

Let $(a, b), (c, d), (e, f) \in A \times A$ such that

$(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

Now, $(a, b)\ R\ (c, d) \Rightarrow a + d = b + c$ ...(i)

$(c, d)\ R\ (e, f)$

$\Rightarrow c + f = d + e$ ...(ii)

Adding (i) and (ii), we get

$(a + d) + (c + f) = (b + c) + (d + e)$

$\Rightarrow a + f = b + e$

$\Rightarrow (a, b)\ R\ (e, f)$

Thus, $(a, b)\ R\ (c, d)$ and $(c, d)\ R\ (e, f)$

$\Rightarrow (a, b)\ R\ (e, f)\ \forall\ (a, b), (c, d), (e, f) \in A \times A$

So, R is transitive on $A \times A$.

$\therefore R$ is an equivalence relation

Equivalence class for $[(2, 5)]$ is $[(1, 4), (2, 5), (3, 6), (4, 7), (5, 8) (6, 9)]$

16. LHS $= \cot^{-1}\left\{\frac{\sqrt{1+\sin x} + \sqrt{1-\sin x}}{\sqrt{1+\sin x} - \sqrt{1-\sin x}}\right\}$

$= \cot^{-1}\left\{\frac{(\sqrt{1+\sin x} + \sqrt{1-\sin x})^2}{(\sqrt{1+\sin x})^2 - (\sqrt{1-\sin x})^2}\right\}$

$= \cot^{-1}\left\{\frac{1+\sin x+1-\sin x+2\cos x}{1+\sin x-1+\sin x}\right\}$

$= \cot^{-1}\left\{\frac{2(1+\cos x)}{2\sin x}\right\}$

$= \cot^{-1}\left\{\frac{1+2\cos^2\frac{x}{2}-1}{2\sin\frac{x}{2}\cdot\cos\frac{x}{2}}\right\} = \cot^{-1}\left(\cot\frac{x}{2}\right) = \frac{x}{2}$

Hence proved.

OR

L.H.S. $= 2\tan^{-1}\left(\frac{1}{5}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right) + 2\tan^{-1}\left(\frac{1}{8}\right)$

$= 2\left[\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)\right] + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\left(\frac{\frac{1}{5}+\frac{1}{8}}{1-\frac{1}{5}\times\frac{1}{8}}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\left(\frac{\frac{13}{40}}{\frac{39}{40}}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\left(\frac{13}{39}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\left(\frac{13}{39}\right) + \sec^{-1}\left(\frac{5\sqrt{2}}{7}\right)$

$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\sqrt{\left(\frac{5}{7}\sqrt{2}\right)^2 - 1}$

$= 2\tan^{-1}\left(\frac{1}{3}\right) + \tan^{-1}\left(\frac{1}{7}\right)$

$$= \tan^{-1}\left(\frac{\frac{2}{3}}{1-\frac{1}{9}}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{1}{7}\right)$$

$$= \tan^{-1}\left(\frac{\frac{3}{4}+\frac{1}{7}}{1-\frac{3}{4}\times\frac{1}{7}}\right) = \tan^{-1}(1) = \frac{\pi}{4}$$

Hence proved.

17. Given $y = x^x$

Taking log on both sides, we have

$\log y = x \log x$ ...(i)

Differentiating (1) w.r.t. x, we get

$$\frac{1}{y}\frac{dy}{dx} = x \times \frac{1}{x} + \log x$$

$$\Rightarrow \frac{y_1}{y} = 1 + \log x \quad \text{...(ii)}$$

Differentiating (2) again w.r.t. x, we have

$$\frac{yy_2 - y_1^2}{y^2} = \frac{1}{x} \Rightarrow yy_2 - y_1^2 = \frac{y^2}{x}$$

$$\Rightarrow y\frac{d^2y}{dx^2} - \left(\frac{dy}{dx}\right)^2 - \frac{y^2}{x} = 0$$

$$i.e.\ \frac{d^2y}{dx^2} - \frac{1}{y}\left(\frac{dy}{dx}\right)^2 - \frac{y}{x} = 0$$

Hence proved.

18. The sample space of the given condition is

$S = \{G_1G_2, G_1B_2, B_1G_2, B_1B_2\}$

where B and G represents boy and girl respectively, and elder child appears first.

Let E : the event that both the children are girls.

F : the event that the youngest child is a girl.

H : the event that atleast one is a girl.

$$\therefore E = \{G_1G_2\} \Rightarrow P(E) = \frac{1}{4}$$

$$F = \{G_1G_2, B_1G_2\} \Rightarrow P(F) = \frac{2}{4} = \frac{1}{2}.$$

$$H = \{G_1B_2, B_1G_2, G_1G_2\} \Rightarrow P(H) = \frac{3}{4}$$

$$\therefore E \cap F = \{G_1G_2\} \Rightarrow P(E \cap F) = \frac{1}{4}$$

$$\therefore E \cap H = \{G_1G_2\} \Rightarrow P(E \cap H) = \frac{1}{4}$$

(i) Required probability = $P(E/F)$

$$= \frac{P(E \cap F)}{P(F)} = \frac{\frac{1}{4}}{\frac{1}{2}} = \frac{1}{2}$$

(ii) Required probability =

$$P(E/H) = \frac{P(E \cap H)}{P(H)} = \frac{1}{3}$$

19. $$\begin{vmatrix} x^2+1 & xy & xz \\ xy & y^2+1 & yz \\ xz & yz & z^2+1 \end{vmatrix} = xyz \begin{vmatrix} x+\frac{1}{x} & y & z \\ x & y+\frac{1}{y} & z \\ x & y & z+\frac{1}{z} \end{vmatrix}$$

[Applying $R_1 \to \frac{1}{x}R_1$, $R_2 \to \frac{1}{y}R_2$, $R_3 \to \frac{1}{z}R_3$]

$$= \frac{xyz}{xyz}\begin{vmatrix} x^2+1 & y^2 & z^2 \\ x^2 & y^2+1 & z^2 \\ x^2 & y^2 & z^2+1 \end{vmatrix}$$ [Applying $C_1 \to xC_1$, $C_2 \to yC_2$, $C_3 \to zC_3$]

$$= \begin{vmatrix} x^2+y^2+z^2+1 & y^2 & z^2 \\ x^2+y^2+z^2+1 & y^2+1 & z^2 \\ x^2+y^2+z^2+1 & y^2 & z^2+1 \end{vmatrix}$$ [Applying $C_1 \to C_1 + C_2 + C_3$]

$$= (x^2+y^2+z^2+1)\begin{vmatrix} 1 & y^2 & z^2 \\ 1 & y^2+1 & z^2 \\ 1 & y^2 & z^2+1 \end{vmatrix}$$

[Taking out $x^2 + y^2 + z^2 + 1$ from C_1]

$$= (1+x^2+y^2+z^2)\begin{vmatrix} 1 & y^2 & z^2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

[Applying $R_2 \to R_2 - R_1$, $R_3 \to R_3 - R_1$]

Now, expanding along C_1, we get

$(1 + x^2 + y^2 + z^2)\,(1(1 - 0) - 0 + 0)$

$= (1 + x^2 + y^2 + z^2) = \text{RHS}$

Hence proved.

20. Let $u = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ and $v = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$

Put $x = \tan\theta$, we get

$$u = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right),\ v = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)$$

$$\Rightarrow u = \tan^{-1}\left\{\frac{\sec\theta - 1}{\tan\theta}\right\},\ v = \sin^{-1}(\sin 2\theta)$$

$$\Rightarrow u = \tan^{-1}\left\{\frac{1-\cos\theta}{\sin\theta}\right\},\ v = 2\theta$$

$$\Rightarrow u = \tan^{-1}\left(\tan\frac{\theta}{2}\right),\ v = 2\theta$$

$\Rightarrow u = \frac{\theta}{2}, v = 2\theta$ ...(i)

Differentiating (i) w.r.t. θ, we get

$\frac{du}{d\theta} = \frac{1}{2}, \frac{dv}{d\theta} = 2$

$\therefore \frac{du}{dv} = \frac{du/d\theta}{dv/d\theta} = \frac{1}{4}$

21. $\frac{dy}{dx} = \frac{x(2\log x + 1)}{\sin y + y\cos y}$

$\Rightarrow (\sin y + y\cos y)dy = x(2\log x + 1)dx$ ...(i)

On integrating (i), we have

$\int(\sin y + y\cos y)dy = \int x(2\log x + 1)dx$

$\Rightarrow -\cos y + y\sin y - (-\cos y)$

$= 2\left[\log x \times \frac{x^2}{2} - \int \frac{1}{x} \times \frac{x^2}{2}dx\right] + \frac{x^2}{2} + C$

$\Rightarrow y\sin y = x^2\log x - \frac{x^2}{2} + \frac{x^2}{2} + C$

$\Rightarrow y\sin y = x^2\log x + C$

Given $y = \frac{\pi}{2}$ when $x = 1$

$\therefore \frac{\pi}{2}\sin\frac{\pi}{2} = 1\cdot\log(1) + C \Rightarrow \frac{\pi}{2} = C$

$\therefore$ Required solution is $y\sin y = x^2\log x + \pi/2$

22. The equation of given lines are

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j})$ and $\vec{r} = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

To find intersection of lines, we must have

$(\hat{i} + \hat{j} - \hat{k}) + \lambda(3\hat{i} - \hat{j}) = (4\hat{i} - \hat{k}) + \mu(2\hat{i} + 3\hat{k})$

$\Rightarrow (3\lambda + 1)\hat{i} + (1 - \lambda)\hat{j} - \hat{k} = (2\mu + 4)\hat{i} + 0\hat{j} + (3\mu - 1)\hat{k}$

$\Rightarrow 3\lambda + 1 = 2\mu + 4,\ 1 - \lambda = 0$ and $-1 = 3\mu - 1$

[On equating coefficients of $\hat{i}$, $\hat{j}$ and $\hat{k}$]

Solving last two of these equations, we get

$\lambda = 1$ and $\mu = 0$.

These values of λ and μ satisfy the equation, $3\lambda + 1 = 2\mu + 4$

So, the given lines intersect.

Putting $\lambda = 1$ in the first line, we get

$\vec{r} = (\hat{i} + \hat{j} - \hat{k}) + (3\hat{i} - \hat{j}) = 4\hat{i} + 0\hat{j} - \hat{k}$

Thus, the coordinates of the point of intersection are $(4, 0, -1)$.

23. The given problem can be formulated as

Maximize $Z = 22x + 18y$

Subject to constraints

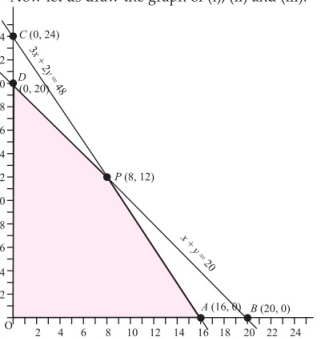
$x + y \le 20$ ...(i)

$360x + 240y \le 5760$

$\Rightarrow 3x + 2y \le 48$ ...(ii)

and $x, y \ge 0$ ...(iii)

Now let us draw the graph of (i), (ii) and (iii).



The shaded region *APDOA* in the above graph is the feasible region.

The corner points of the feasible region are $A(16, 0)$, $P(8, 12)$, $D(0, 20)$ and $O(0, 0)$.

Now,

Corner points	$Z = 22x + 18y$
(16, 0)	352
(8, 12)	392
(0, 20)	360
(0, 0)	0

We see that the point (8, 12) is giving the maximum value of Z.

Hence the dealer should purchase 8 electronic sewing machines and 12 manually operated sewing machines to obtain the maximum profit under the given condition.

24. Let E_1, E_2, E_3, E_4 and A be the event defined as below

E_1 = missing card is a heart card.

E_2 = missing card is a spade card.

E_3 = missing card is a club card.

E_4 = missing card is a diamond card.

A = drawing three spade cards from the remaining cards.

Now, $P(E_1) = P(E_2) = P(E_3) = P(E_4) = \frac{13}{52} = \frac{1}{4}$

$P(A/E_2) = \frac{{}^{12}C_3}{{}^{51}C_3}$

$P(A/E_1) = P(A/E_3) = P(A/E_4) = \frac{{}^{13}C_3}{{}^{51}C_3}$

$$P(E_2/A)=\frac{P(E_2)P(A/E_2)}{P(A/E_1)P(E_1)+P(A/E_2)P(E_2)+(A/E_3)P(E_3)+P(A/E_4)P(E_4)}$$

$$=\frac{\frac{1}{4}\times\frac{{}^{12}C_3}{{}^{51}C_3}}{\frac{1}{4}\times\frac{{}^{12}C_3}{{}^{51}C_3}+\frac{3}{4}\times\frac{{}^{13}C_3}{{}^{51}C_3}}=\frac{{}^{12}C_3}{{}^{12}C_3+3\times{}^{13}C_3}$$

$$=\frac{220}{220+858}=\frac{110}{539}$$

OR

Total no. of bulbs = 15

No. of defective bulbs = 5

$P(\text{Defective bulb})=\frac{5}{15}=\frac{1}{3}$

No. of non-defective bulbs = 10

$P(\text{Non-defective bulb})=\frac{10}{15}=\frac{2}{3}$

Let X be a random variable that shows the number of defective bulbs in a draw of 4 bulbs

$\therefore$ X can take values 0, 1, 2, 3 and 4.

Now, using Binomial distribution,

$$P(X=0)={}^4C_0\left(\frac{1}{3}\right)^0\left(\frac{2}{3}\right)^4=\frac{16}{81}$$

$$P(X=1)={}^4C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^3=\frac{32}{81}$$

$$P(X=2)={}^4C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^2=\frac{24}{81}$$

$$P(X=3)={}^4C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)=\frac{8}{81}$$

$$P(X=4)={}^4C_4\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^0=\frac{1}{81}$$

$\therefore$ The probability distribution table is as follows:

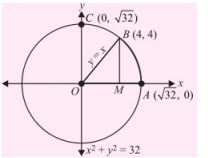
X	0	1	2	3	4
$P(X)$	$\frac{16}{81}$	$\frac{32}{81}$	$\frac{24}{81}$	$\frac{8}{81}$	$\frac{1}{81}$

and Mean, $\bar{x}=\sum XP(X)$

$$=\frac{0+32+48+24+4}{81}=\frac{108}{81}=\frac{4}{3}$$

25. Given $y=x$ ...(i) and $x^2+y^2=32$ ...(ii)

The region enclosed by $y=x$, $x^2+y^2=32$ and x-axis is shown below:



$\therefore$ Required area = $OBAO$

= Area of ΔOBM + Area of region $BMAB$

$$=\int_0^4 y\,dx+\int_4^{32}y\,dx=\int_0^4 x\,dx+\int_4^{\sqrt{32}}\sqrt{32-x^2}\,dx$$

$$=\left|\frac{x^2}{2}\right|_0^4+\left[\frac{x}{2}\sqrt{32-x^2}+\frac{32}{2}\sin^{-1}\left(\frac{x}{\sqrt{32}}\right)\right]_4^{\sqrt{32}}$$

$$=8+16\times\frac{\pi}{2}-8-16\times\frac{\pi}{4}=\frac{16\pi}{4}=4\pi \text{ sq. units}$$

26. The equation of the plane passing through (2, 5, – 3) is

$a(x-2)+b(y-5)+c(z+3)=0$ ...(i)

If plane (i) passes through the points (–2, –3, 5) and (5, 3, – 3), then

$-4a-8b+8c=0\Rightarrow -a-2b+2c=0$ ...(ii)

and $3a-2b+0=0$ ...(iii)

Solving (ii) and (iii), we get

$$\therefore\ \frac{a}{-4}=\frac{b}{-6}=\frac{c}{-2-6}$$

$$\Rightarrow \frac{a}{2}=\frac{b}{3}=\frac{c}{4}=\lambda(\text{say})$$

Substituting the values of a, b, c in (i), we get

$2\lambda(x-2)+3\lambda(y-5)+4\lambda(z+3)=0$

$\Rightarrow 2x+3y+4z=7$ which is the equation of the plane passing through the given three points.

Now the distance of (7, 2, 4) from this plane is given by

$$\left|\frac{14+6+16-7}{\sqrt{4+9+16}}\right|=\frac{29}{\sqrt{29}}=\sqrt{29}\text{ units}$$

OR

The given line is

$\vec{r}=(2\hat{i}-\hat{j}+2\hat{k})+\lambda(3\hat{i}+4\hat{j}+2\hat{k})$ ...(i)

and the plane is, $\vec{r}\cdot(\hat{i}-\hat{j}+\hat{k})=5$ ...(ii)

Solving (i) & (ii), we get

$[(2\hat{i}-\hat{j}+2\hat{k})+\lambda(3\hat{i}+4\hat{j}+2\hat{k})]\cdot(\hat{i}-\hat{j}+\hat{k})=5$

$\Rightarrow (2+3\lambda)\cdot(1)+(4\lambda-1)(-1)+(2\lambda+2)(1)=5$

$\Rightarrow 2+3\lambda-4\lambda+1+2\lambda+2=5\Rightarrow\lambda=0$

$\therefore$ Point of intersection of (i) and (ii) is, (2, –1, 2)

Now, the distance between (2, –1, 2) and (–1, –5, –10) is,

$$\sqrt{(2+1)^2+(-1+5)^2+(2+10)^2}$$

$$=\sqrt{9+16+144}=13\text{ units}$$

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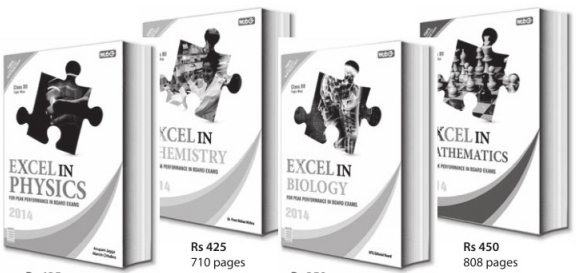

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27. We have,

$3x + 2y + z = 1000$ ...(i)

$4x + y + 3z = 1500$ ...(ii)

$x + y + z = 600$ ...(iii)

$\therefore$ Now $|A| = \begin{vmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{vmatrix} = -5 \neq 0$

$\therefore$ A is invertible

$\therefore$ $AX = B \Rightarrow A^{-1}(AX) = A^{-1}B \Rightarrow X = A^{-1}B$

where $A = \begin{pmatrix} 3 & 2 & 1 \\ 4 & 1 & 3 \\ 1 & 1 & 1 \end{pmatrix}$ and $B = \begin{pmatrix} 1000 \\ 1500 \\ 600 \end{pmatrix}$

Cofactor matrix of $A = \begin{bmatrix} -2 & -1 & 3 \\ -1 & 2 & -1 \\ 5 & -5 & -5 \end{bmatrix}$

$\therefore$ $A^{-1} = \dfrac{\text{adj}(A)}{|A|} = \dfrac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}$

Now, $X = A^{-1}B$

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \frac{-1}{5}\begin{bmatrix} -2 & -1 & 5 \\ -1 & 2 & -5 \\ 3 & -1 & -5 \end{bmatrix}\begin{bmatrix} 1000 \\ 1500 \\ 600 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} x \\ y \\ z \end{bmatrix} = -\frac{1}{5}\begin{bmatrix} -2000 - 1500 + 3000 \\ -1000 + 3000 - 3000 \\ 3000 - 1500 - 3000 \end{bmatrix} = \begin{bmatrix} 100 \\ 200 \\ 300 \end{bmatrix}$$

$\Rightarrow x = 100,\ y = 200,\ z = 300$

Hence the money awarded for discipline, politeness and punctuality are ₹ 100, ₹ 200 and ₹ 300.

Also, the determinant A is non-zero. Therefore x, y and z will have unique solution.

Honesty should also be awarded.

28. $I = \int_0^{\pi/2} \dfrac{x \sin x \cos x}{\sin^4 x + \cos^4 x}\,dx$

$$= \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right)\sin\left(\frac{\pi}{2} - x\right)\cos\left(\frac{\pi}{2} - x\right)}{\sin^4\left(\frac{\pi}{2} - x\right) + \cos^4\left(\frac{\pi}{2} - x\right)}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx - \int_0^{\pi/2} \frac{x \sin x \cos x}{\cos^4 x + \sin^4 x}$$

$$\Rightarrow 2I = \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{\cos^4 x + \sin^4 x}\,dx$$

$$= \frac{\pi}{2}\int_0^{\pi/2} \frac{\sin x \cos x}{(1 - \sin^2 x)^2 + \sin^4 x}\,dx$$

Put $t = \sin^2 x \Rightarrow \dfrac{dt}{dx} = 2 \sin x \cos x$

$\Rightarrow$ $dt = 2\sin x \cos x\,dx \Rightarrow \sin x \cos x\,dx = dt/2$

when $x = 0 \Rightarrow t = 0$ and when $x = \dfrac{\pi}{2} \Rightarrow t = 1$

$$\therefore \quad 2I = \frac{\pi}{2}\int_0^1 \frac{dt/2}{(1-t)^2 + t^2} = \frac{\pi}{4}\int_0^1 \frac{dt}{1 - 2t + t^2 + t^2}$$

$$= \frac{\pi}{4}\int_0^1 \frac{dt}{2t^2 - 2t + 1} = \frac{\pi}{4}\int_0^1 \frac{dt}{2(t^2 - t + 1/2)}$$

$$= \frac{\pi}{8}\int_0^1 \frac{dt}{t^2 - t + \frac{1}{4} - \frac{1}{4} + \frac{1}{2}} = \frac{\pi}{8}\int_0^1 \frac{dt}{\left(t - \frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2}$$

$$\Rightarrow 2I = \frac{\pi}{8}\frac{1}{\left(\frac{1}{2}\right)}\tan^{-1}\frac{t - \frac{1}{2}}{\frac{1}{2}}\Bigg]_0^1 = \frac{\pi}{4}\tan^{-1}(2t - 1)\Big]_0^1$$

$$\Rightarrow 2I = \frac{\pi}{4}[\tan^{-1}(1) - \tan^{-1}(-1)] = \frac{\pi}{4}\left[\frac{\pi}{4} - \left(-\frac{\pi}{4}\right)\right]$$

$$\Rightarrow 2I = \frac{\pi}{4}\left(\frac{\pi}{4} + \frac{\pi}{4}\right) = \frac{\pi^2}{8} \quad \Rightarrow I = \frac{\pi^2}{16}$$

29. Let r and h be the radius and height of the cylindrical can respectively.

Therefore, the total surface area of the closed cylinder is given by

$S = 2\pi rh + 2\pi r^2 = 2\pi r(r + h)$ ...(i)

Given volume of the can = 128π cm^3

Also volume $(V) = \pi r^2 h$ ...(ii)

$\Rightarrow$ $\pi r^2 h = 128\pi \Rightarrow h = \dfrac{128}{r^2}$ ...(iii)

Putting the value of h in equation (i), we have

$$S = 2\pi r\left(r + \frac{128}{r^2}\right) = 2\pi r^2 + \frac{256}{r}\pi$$ ...(iv)

Differentiating (iv) w.r.t. r, we have

$$\frac{dS}{dr} = 4\pi r + 256 \times (-r^{-2})\pi = 4\pi r - \frac{256\pi}{r^2}$$

Substituting $\dfrac{dS}{dr} = 0$ for critical points, we have

$$\Rightarrow 4\pi r - \frac{256\pi}{r^2} = 0 \Rightarrow r^3 = 64 \Rightarrow r = 4 \text{ cm}$$

Now $\dfrac{dS}{dr} = 4\pi r - \dfrac{256\pi}{r^2}$

$$\Rightarrow \quad \frac{d^2S}{dr^2} = 4\pi - 256\pi(-2r^{-3}) = 4\pi + \frac{512}{r^3}\pi$$

$$\therefore \quad \frac{d^2S}{dr^2} > 0$$

Thus the total surface area of the cylinder is minimum when $r = 4$.

From equation (iii), we have

$$h = \frac{128}{r^2} = \frac{128}{16} = 8 \Rightarrow \quad h = 8 \text{ cm}$$

Thus the dimensions of the cylinder is radius = 4 cm and height = 8 cm. ■■

PRACTICE PAPER 2014

ISI

(Indian Statistical Institute)

Exam on 11th May 2014

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PART-A

Multiple Choice Test

1. Let $A(2, 2)$ and $B(7, 7)$ be points in the plane. Define R as the region in the first quadrant consisting of those points C such that ΔABC is an acute triangle. What is the closest integer to the area of the region R?

(a) 25 (b) 39 (c) 51 (d) 60

2. Let $P(x) = (x - 1)(x - 2)(x - 3)$. For how many polynomials $Q(x)$ does there exist a polynomial $R(x)$ of degree 3 such that $P(Q(x)) = P(x)\cdot R(x)$?

(a) 19 (b) 22 (c) 24 (d) 27

3. For all positive integers n less than 2002, let

$$a_n = \begin{cases} 11, \text{ if } n \text{ is divisible by 13 and 14} \\ 13, \text{ if } n \text{ is divisible by 14 and 11} \\ 14, \text{if } n \text{ is divisible by 11and 13} \\ 0, \text{otherwise} \end{cases}$$

The value of $\sum_{n=1}^{2001} a_n$ is

(a) 448 (b) 486 (c) 1560 (d) 2001

4. Mrs. Sharma gave an exam in a mathematics class of five students. She entered the scores in random order into a spreadsheet, which recalculated the class average after each score was entered. Mrs. Sharma noticed that after each score was entered, the average was always an integer. The scores (listed in ascending order) were 71, 76, 80, 82, and 91. What was the last score Mrs. Sharma entered?

(a) 71 (b) 76 (c) 80 (d) 82

5. The number of positive integers 'n' such that $n + 9$, $16n + 9$, $27n + 9$ are all perfect squares is

(a) 1 (b) 2 (c) 3 (d) 4

6. A convex quadrilateral $ABCD$ with area 2002 sq.units contains a point P in its interior such that $PA = 24$, $PB = 32$, $PC = 28$ and $PD = 45$. The perimeter of $ABCD$ (in units) is

(a) $4\sqrt{2002}$ (b) $2(48 + \sqrt{2002})$

(c) $2\sqrt{8633}$ (d) $4(36 + \sqrt{113})$

7. The number of real ordered pairs (x, y) satisfying $x^2 + x = y^4 + y^3 + y^2 + y$ is

(a) 3 (b) 4 (c) 5 (d) 6

8. There are 500 students in a school. Two thirds of the students who do not wear spectacles do not bring lunch, three - quarters of the students who do not bring lunch do not wear spectacles. Altogether 60 spectacled students bring lunch too. The number of students who do not wear spectacles and do not bring lunch is

(a) 180 (b) 140 (c) 240 (d) 320

9. Given two sides a and b of a triangle, and it is known that the medians drawn to the given sides intersect at a right angle. What are the conditions for the triangle to exist?

(a) $\frac{1}{2} < \frac{a}{b} < 2$ (b) $\frac{1}{4} < \frac{a}{b} < 4$

(c) $\frac{1}{2\sqrt{2}} < \frac{a}{b} < 2\sqrt{2}$ (d) $\frac{1}{4\sqrt{2}} < \frac{a}{b} < 4\sqrt{2}$

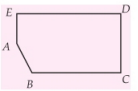
10. The number of real ordered triplets (x, y, z) satisfying the equations,

$$\frac{4x^2}{1+4x^2} = y, \frac{4y^2}{1+4y^2} = z, \frac{4z^2}{1+4z^2} = x \text{ is}$$

(a) 1 (b) 2 (c) 3 (d) 4

11. A point P is selected at random from the interior of the pentagon with vertices $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 0)$ and $E = (0, 4)$. What is the probability that $\angle APB$ is obtuse?

*** Alok Kumar is a winner of INDIAN NATIONAL MATHEMATICS OLYMPIAD (INMO-91). He trains IIT and Olympiad aspirants.**



(a) 1/5 (b) 1/4
(c) 5/16 (d) 3/8

12. Inside an equilateral triangle ABC an arbitrary point P is taken from which the perpendiculars PD, PE and PF are dropped onto BC, CA and AB respectively. The value of $\frac{PD+PE+PF}{BD+CE+AF}$ is

(a) $3\sqrt{3}$ (b) $\sqrt{3}$ (c) $\frac{1}{\sqrt{3}}$ (d) $\frac{1}{3\sqrt{3}}$

13. A set S of points in the xy-plane is symmetric about the origin, both coordinate axes, and the line $y = x$. If (2, 3) is in S, what is the smallest number of points in S?

(a) 2 (b) 4 (c) 8 (d) 16

14. Let ΔXOY be a right-angled triangle with $\angle XOY = 90°$. Let M and N be the midpoints of legs OX and OY, respectively. Given that XN = 19 and YM = 22, XY is

(a) 24 (b) 26 (c) 28 (d) 30

15. ABC is an isosceles triangle with $\angle B = \angle C = 78°$, D and E are points on AB, AC respectively, such that $\angle BCD = 24°$ and $\angle CBE = 51°$. $\angle BED$ is

(a) 48° (b) 36° (c) 12° (d) 40°

16. A is the set of three digit numbers formed using the digits 1, 2, 3, ..., 9, using each digit at the most once. Also every number in A has exactly one even digit. Find the sum of all the numbers in A.

(a) 133200 (b) 96100
(c) 199800 (d) 77350

17. In a railway reservation centre, three digit token numbers ranging from 001 to 999 are issued to people. A single - digit display is flashed three times to show the three digit (from left to right) of the number to be called out. Due to some flaw, some lines in the display are permanently missing, but others work perfectly fine. If the numbers flashed look like ⌊⌋, ⌋⌊ ⌊ in that order. then how many people have a reason to believe that it might be their number?

(a) 5 (b) 30 (c) 179 (d) 209

18. Find the number of points with integer coordinates which are at a distance of less than 5 units from the point (1, 2) on the xy plane.

(a) 61 (b) 69 (c) 81 (d) 105

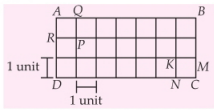
19. Find the area of the shaded region in the figure, if the side of the largest square is 4 cm and the other square are the largest square that can be drawn inside the respective quarter circles.

(a) 6.4 cm^2 (b) 8 cm^2
(c) 4.4 cm^2 (d) 12 cm^2

20. A computer program when fed with a real number x, generates another number y such that y is the greatest integer less than or equal to x. Amisha inputs the values $\sqrt{1}, \sqrt{2}, \sqrt{3}, \sqrt{4}, ..., \sqrt{100}$ sequentially and finds the product of all the numbers that the computer gives as output. The product is :

(a) $2^{87} \times 3^{59} \times 5^{13} \times 7^{14}$
(b) $2^{88} \times 3^{58} \times 5^{12} \times 7^{15}$
(c) $2^{81} \times 3^{59} \times 5^{12} \times 7^{17}$
(d) $2^{79} \times 3^{61} \times 5^{21} \times 7^{12}$

21. An ant has to travel from point C to A along the gridlines. It gets lost on its way but manages to reach A by travelling 12 units in all. Its first two moves are CM and MK and its last two moves are PQ and QA. In how many ways could the ant have travelled from C to A, if it never travels downwards?



(a) 336 (b) 56 (c) 5 (d) 30

22. Each cell in a 4 × 2 grid is to be painted with one colour out four colour red, blue, green and yellow, such that no two square painted with the same colour are in the same row or in the same column. Find the number of ways to do this.

(a) 4! × 3! (b) 23 × 4!
(c) 9 × 4! (d) 4 × 4!

23. The third digit from the right of the number 101^{76} is

(a) 1 (b) 0 (c) 6 (d) 9

Direction (24-25) : Answer the questions on the basis of the information given below :

$f(x) = f(x + 1) + f(x + 2)$

$g(x) = \dfrac{x}{x-1}$

$f(f(x)) = f^2(x)$, $f(f(f(x))) = f^3(x)$ and so on.

24. Find the value of $f(0)$, if $f(1) = 2/3$ and $f(g^{22}(3)) = \dfrac{1}{3}$

(a) 1 (b) 1/3 (c) –1/3 (d) 0

25. Find x, if $g^{2n+2}(x) = \dfrac{1}{f(7)}$, where n is a natural number. (Use data from the previous question.)

(a) 2/3 (b) 0 (c) 3/2 (d) 1

26. If $\log_N 2M = 1$ and $\dfrac{1}{2}\log_M 0.25 + 2\log_5 N = \log_M N + 1$, then find the value of N.

(a) 10 (b) 1/10 (c) 12 (d) 2

27. Consider a polynomial $p(x) = a_3x^3 + a_2x^2 + a_1x + a_0$ such that $p(0)$ and $p(1)$ are odd. How many integer roots does $p(x)$ have?

(a) 0 (b) 1 (c) 2 (d) 3

28. $S = \{1, 2, ..., 5n\}$, where n is a natural number. If R is a subset of S such that the average of no two elements of R is n or an integral multiple of n, then the maximum number of elements in R is

(a) $4n - 1$ (b) $2n$ (c) $3n$ (d) $3n - 1$

29. AB and CD are perpendicular chords of a circle of radius 10 cm, which are at a distance 6 cm and $5\sqrt{2}$ m respectively from the centre of the circle. Find a possible value of $l(AD) \times l(BC)$.

(a) $140\sqrt{2}\,\text{cm}^2$ (b) $20\sqrt{3}\,\text{cm}^2$

(c) $80\sqrt{3}\,\text{cm}^2$ (d) $200\sqrt{2}\,\text{cm}^2$

30. A regular polygon with an even number of sides is inscribed in a circle. 126 of the diagonals of the polygon do not pass through the centre of the circle in which it is inscribed. Find the number of sides of the polygon.

(a) 22 (b) 14 (c) 16 (d) 18

PART-B

Short - Answer Type Test

31. Find a point P inside a triangle ABC such that the product of its distances from the sides is a maximum.

32. Find all triplets (x, y, z) of positive integers satisfying $2^x + 2^y + 2^z = 2336$.

33. Let 'a' be a real number such that $a^5 - a^3 + a = 2$. Prove that $3 < a^6 < 4$.

34. Suppose a and b are two positive real numbers such that the roots of the cubic equation $x^3 - ax + b = 0$ are all real. If α is a root of this cubic with minimal absolute value prove that $\dfrac{b}{a} < \alpha \le \dfrac{3b}{2a}$.

35. Let ABC be a triangle and h_a be the altitude through A. Prove that $(b + c)^2 \ge a^2 + 4h_a^2$. (As usual a, b, c denote the sides BC, CA, AB respectively.)

36. Let a, b, c be positive real numbers. Show that there is a triangle with sides a, b, c if and only if there exist real numbers x, y, z such that

$$\frac{y}{z} + \frac{z}{y} = \frac{a}{x},\ \frac{z}{x} + \frac{x}{z} = \frac{b}{y},\ \frac{x}{y} + \frac{y}{x} = \frac{c}{z}.$$

37. Find all positive integers n that have exactly 16 positive integral divisors $d_1, d_2, ..., d_{16}$ such that $1 = d_1 < d_2 < ... < d_{16} = n$, $d_6 = 18$ and $d_9 - d_8 = 17$.

38. Let ABC be a non-obtuse triangle such that $AB > AC$ and $\angle B = 45°$. Let O and I denote the circumcentre and incentre ΔABC respectively. Suppose that $\sqrt{2}OI = AB - AC$. Determine all the possible values of $\sin\angle BAC$.

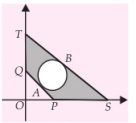
39. A triangle ABC has positive integer sides, $\angle A = 2\angle B$ and $\angle C > 90°$. Find the minimum length of the perimeter of ABC.

40. Let m, n be natural numbers such that $A = \dfrac{(m+3)^n + 1}{3m}$ is an integer. Prove that A is odd.

SOLUTIONS

1. (c) : For ΔABC to be acute, all angles must be acute. For $\angle A$ to be acute. Point C must lie above the line passing through A and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $P(4, 0)$ and $Q(0, 4)$. For $\angle B$ to be acute, point C must lie below the line through B and perpendicular to $\overline{AB}$. The segment of that line in the first quadrant lies between $S(14, 0)$ and $T(0, 14)$. For $\angle C$ to be acute, point C must lie outside the circle U that has $\overline{AB}$ as a diameter. Let O denote the origin. Region R, shaded below, has area equal to

Area(ΔOST) – Area(ΔOPQ) – Area (circle U)



$$= \frac{1}{2}\cdot 14^2 - \frac{1}{2}\cdot 4^2 - \pi\left(\frac{\sqrt{50}}{2}\right)^2$$

$$= 90 - \frac{25}{2}\pi = 51$$

2. (b): $P(x)$ has degree 6, so $Q(x)$ must have degree 2. Therefore Q is uniquely determined by the ordered triplet $(Q(1), Q(2), Q(3))$. Then $x = 1$, 2, or 3. We have $0 = P(x)\cdot R(x) = P(Q(x))$. It follows that $(Q(1), Q(2), Q(3))$ is one of the 27 ordered triplets (i, j, k), where i, j, and k can be chosen from the set {1, 2, 3}. However, the choices (1, 1, 1), (2, 2, 2), (3, 3, 3), (1, 2, 3) and (1, 2, 3) lead to polynomials $Q(x)$ defined by $Q(x) = 1$, 2, 3, x and $4 - x$, respectively, all of which have degree less than 2. The other 22 choices for $Q(1)$, $Q(2)$, $Q(3)$ yield non - collinear points, so in each case $Q(x)$ is a quadratic polynomial.

3. (a): Since 2002 = 11, 13, 14, we have

$$a_n = \begin{cases} 11, \text{ if } n = 13\cdot 14\cdot i, \text{ where } i = 1,2,.....,10; \\ 13, \text{ if } n = 14\cdot 11\cdot j, \text{ where } j = 1,2.....,12, \\ 14, \text{ if } n = 11\cdot 13\cdot k, \text{ where } k = 1,2,.....,13, \\ 0, \text{ otherwise} \end{cases}$$

Hence $\sum_{n=1}^{2001} a_n = 11\cdot 10 + 13\cdot 12 + 14\cdot 13 = 448.$

4. (c): Note that the integer average condition means that the sum of the score of the first n students is a multiple of n. The scores of the first two students must be both even or both odd and the sum of the scores of the first three students must be divisible by 3. The remainders when 71, 76, 80, 82 and 91 are divided by 3 are 2, 1, 2, 1 and 1, respectively. Thus the only sum of the three scores divisible by 3 is 76 + 82 + 91 = 249, so the first two scores entered are 76 and 82 (in some order), and the third scores is 91. Since 249 is 1 larger than a multiple of 4, the fourth score must be 3 larger than a multiple of 4, and the only possibility is 71, leaving 80 as the scores of the fifth student.

5. (a): Let $n + 9 = a^2$, $16n + 9 = b^2$, $27n + 9 = c^2$, where n, a, b, c are positive integers.

$16a^2 = 16n + 144$

$b^2 = 16n + 9$

Subtracting, we get $16a^2 - b^2 = 135$

$\Rightarrow$ $(4a - b)(4a + b) = 135 = 1 \times 135, 3 \times 45, 5 \times 27, 9 \times 15$

$\Rightarrow$ $a = 17, 6, 4, 3$

$\Rightarrow$ $n = 280, 27, 7, 0$

$27n + 9 = 9(3n + 1) = 9(841), 9(22), 9(1)$

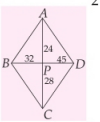
$9(841) = 3^2 \times 29^2 = (3 \times 29)^2 = 87^2$

This is the only square among them apart from the trivial case $n = 0$

When $n = 280$, $n + 9 = 289 = 17^2$

$16n + 9 = 67^2$, $27n + 9 = 87^2$.

6. (d): We have, area $(ABCD) \le \frac{1}{2} AC\cdot BD$



with equality if and only if $AC \perp BD$. Since

$$2002 = \text{Area}(ABCD) \le \frac{1}{2} AC\cdot BD$$

$$\le \frac{1}{2}(AP + PC)\cdot(BP + PD) = \frac{52\cdot 77}{2} = 2002$$

It follows that the diagonals AC and BD are perpendicular and intersect at P.

Thus, $AB = \sqrt{24^2 + 32^2} = 40$,

$BC = \sqrt{28^2 + 32^2} = 4\sqrt{113}, CD = \sqrt{28^2 + 45^2} = 53$ and

$DA = \sqrt{45^2 + 24^2} = 51$. The perimeter of $ABCD$ is therefore $144 + 4\sqrt{113} = 4(36 + \sqrt{113})$

7. (d): $x^2 + x = y^4 + y^3 + y^2 + y$

$4x^2 + 4x + 1 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$

$(2x + 1)^2 = 4y^4 + 4y^3 + 4y^2 + 4y + 1$

Now,

$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2y^2 + y)^2 + (3y + 1)(y + 1)$ ...(1)

$4y^4 + 4y^3 + 4y^2 + 4y + 1 = (2y^2 + y + 1)^2 - y(y - 2)$ ...(2)

Consider $(3y + 1)(y + 1)$

This is positive when $y < -1$ or $y > 0$

$\therefore$ $4y^4 + 4y^3 + 4y^2 + 4y + 1 > (2y^2 + y)^2$

$y(y - 2) > 0$ for $y < 0$, $y < 2$

$4y^4 + 4y^3 + 4y^2 + 4y + 1 < (2y^2 + y + 1)^2$

$\therefore$ $(2y^2 + y)^2 < 4y^4 + 4y^3 + 4y^2 + 4y + 1 < (2y^2 + y + 1)^2$

This lies betwween two consecutive squares.

$\Rightarrow$ it cannot be a square.

But we must check for $y = -1$, 0, 1, 2.

When $y = -1$, $y^4 + y^3 + y^2 + y = 1 - 1 + 1 = 0$

$\therefore\ x^2 + x = 0 \Rightarrow x(x+1) = 0,\ x = 0, x = -1$

$\therefore\quad x = 0,\ y = -1\ ;\ x = -1,\ y = -1$

When $y = 0$, $x^2 + x = 0$, $x = 0$, $x = 0$, $x = -1$, we get (0, 0), (–1,0).

When $y = 1$, $y^4 + y^3 + y^2 + y = 4$

$x^2 + x = 4$ no integer solution.

When $y = 2$, $y^4 + y^3 + y^2 + y = 16 + 8 + 4 + 2 = 30$

$x^2 + x = 30$, $x(x + 1) = 30x = 5, -6$

Solutions are (0, –1), (–1, –1), (0, 0), (–1, 0), (5, 2), (–6, 2)

8. (c) : Let the number of students wearing spectacles = a

Let the number of students bringing lunch = b.

The number of students who either wear spectacles or bring lunch or both = $a + b - 60$.

The number of students who do not wear spectacles and do not bring lunch =

$500 - (a + b - 60) = 560 - a - b$

This is also equal to $\frac{2}{3}(500 - a)$ or $\frac{3}{4}(500 - b)$

$\Rightarrow \frac{2}{3}(500 - a) = \frac{3}{4}(500 - b) = 560 - a - b$

There are two unknowns a, b to be evaluated.

$\frac{2}{3}(500 - a) = \frac{3}{4}(500 - b)$

$8(500 - a) = 9(500 - b)$

$8a - 9b + 500 = 0$ ... (1)

Also $3(500 - b) = 4(560 - a - b)$

$1500 - 3b = 2240 - 4a - 4b$

$4a + b = 740$ ... (2)

From (1) and (2), $b = 180$, $a = 140 \Rightarrow a + b = 320$

No. of students who do not wear spectacles and do not bring lunch = 560 – 320 = 240

9. (a) : In the triangle ABC, the medians AD and BE intersect at a point O, $AC = b$ and $BC = a$. Let us find $AB = c$.

Let $OD = x$ and $OE = y$. Taking advantage of the property of medians we find from the triangles AOB, BOD and AOE that

$4x^2 + y^2 = \frac{b^2}{4}, 4x^2 + 4y^2 = c^2, 4x^2 + 16y^2 = a^2$

Eliminating x and y we obtain $c^2 = \frac{a^2 + b^2}{5}$

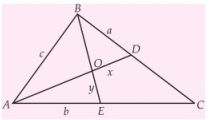
The conditions for existence of a triangle with sides a, b and c takes the form

$5(a + b)^2 > a^2 + b^2$, $5(a - b)^2 < a^2 + b^2$.

The first inequality is obviously fulfilled for any a and b, and the second one is transformed into the relation:

$a^2 - \frac{5}{2}ab + b^2 < 0$.

Solving this inequality with respect to $\frac{a}{b}$ we finally obtain $\frac{1}{2} < \frac{a}{b} < 2$.



10. (a) : The values of x, y, z are real.

Clearly $x = 0$, $y = 0$, $z = 0$ do not satisfy the equation

$\Rightarrow\quad x \neq 0,\ y \neq 0,\ z \neq 0$

Multiply all the equations, we have

$$\frac{64x^2y^2z^2}{(1+4x^2)(1+4y^2)(1+4z^2)} = xyz$$

$\Rightarrow\quad 64xyz = (1 + 4x^2)(1 + 4y^2)(1 + 4z^2)$ ... (1)

Clearly $(1 - 2x)^2 \geq 0 \quad \Rightarrow \quad 1 + 4x^2 \geq 4x$

$1 + 4y^2 \geq 4y \ \Rightarrow\ 1 + 4z^2 \geq 4z$

$\Rightarrow (1 + 4x^2)(1 + 4y^2)(1 + 4z^2) \geq 64xyz$

The equality holds when $x = y = z$,

Now $\frac{4x^2}{1 + 4x^2} = y$

$\Rightarrow\quad 4x^2 = y(1 + 4x^2) \quad \Rightarrow \quad 4x^2 = x(1 + 4x^2)$

$x \neq 0 \quad \Rightarrow \quad 4x = 1 + 4x^2$

or $(1 - 2x)^2 = 0$ or $x = \frac{1}{2}, y = \frac{1}{2}, z = \frac{1}{2}$

$\therefore$ Only one triplet is possible.

11. (c) : Since $\angle APB = 90°$ if and only if P lies on the semicircle with centre (2, 1) and radius $\sqrt{5}$, the angle is obtuse if and only if the point P lies inside this semicircle. The semicircle lies entirely inside the pentagon, since the distance, 3, from (2, 1) to $\overline{DE}$ is greater than the radius of OB the circle. Thus, the probability that the angles is obtuse is the ratio of the area of the semicircle to the area of the pentagon.

Let $O = (0, 0)$, $A = (0, 2)$, $B = (4, 0)$, $C = (2\pi + 1, 0)$, $D = (2\pi + 1, 4)$ and $E = (0, 4)$.

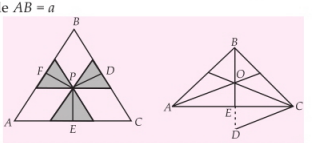
Then the area of the pentagon is

$[ABCDE] = [OCDE] - [OAB] = 4 \cdot (2\pi + 1) - \frac{1}{2}(2 \cdot 4) = 8\pi$

and the area of the semicircle is $\frac{1}{2}\pi(\sqrt{5})^2 = \frac{5}{2}\pi$.

The probability is $\frac{5/2\pi}{8\pi} = \frac{5}{16}$.

12. (c) : Draw through the point P three straight lines parallel to the sides of the triangle. The three triangles thus formed (they are shaded in the figure) are also equilateral, and the sum of their sides is equal to the side $AB = a$



Consequently, the sum of their altitudes is equal to the altitude of ΔABC and hence

$$PD + PE + PF = \frac{a\sqrt{3}}{2}$$

The sum $BD + CE + AF$ is equal to the sum of the sides of the shaded triangle added to the sum of the halves of these sides and thus,

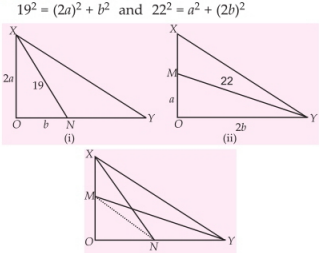
$$BD + CE + AF = \frac{3}{2}a$$

Consequently, $\dfrac{PD + PE + PF}{BD + CE + AF} = \dfrac{1}{\sqrt{3}}$

13. (d): The set S is symmetric about the line $y = x$ and contains (2, 3), so it must also contain (3, 2). Also S is symmetric about the x-axis, so it must contain (2, –3) and (3, –2). Finally, since S is symmetric about the y-axis, it must contain (–2, 3),(–3, 2), (–2, –3) and (–3, –2). Since the resulting set of 8 points is symmetric about both coordinate axes, it is also symmetric about the origin.

14. (b): Let $OM = a$ and $ON = b$, Then

$19^2 = (2a)^2 + b^2$ and $22^2 = a^2 + (2b)^2$



Hence, $5(a^2 + b^2) = 19^2 + 22^2 = 845$

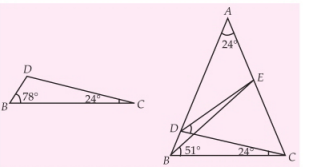
It follows that $MN = \sqrt{a^2 + b^2} = \sqrt{169} = 13$

Since ΔXOY is similar to ΔMON

$$\therefore \quad \frac{OX}{MO} = \frac{XY}{MN} \Rightarrow \frac{2a}{a} = \frac{XY}{13}$$

$\Rightarrow XY = 26.$

15. (c) : ΔABC is isosceles in which $AB = AC$, $\angle ABC = \angle ACB$



In ΔBDC, $\angle BDC = 180° - (78° + 24°) = 78°$

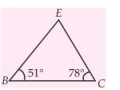
$\Rightarrow \quad BC = CD \qquad \ldots (1)$

Consider triangle BEC.

$\angle BEC = 180° - (51° + 78°) = 51°$

$\Rightarrow \quad BC = EC \qquad \ldots (2)$

From (1) and (2) we get $CD = EC$.



$$\angle DCE = 78° - 24° = 54 \quad \therefore \angle DEC = \frac{180° - 54°}{2} = 63$$

$$\therefore \quad \angle DEB = \angle DEC - \angle BEC = 63° - 51° = 12°.$$

16. (a) : Let 'abc' be a number in A.

Then all possible arrangements of 'a', 'b' and 'c' will also be numbers in A.

Sum of all possible arrangements of 'a', 'b' and 'c' is

$200(a + b + c) + 20(a + b + c) + 2\ (a + b + c)$

$= 222(a + b + c)$

All the digits in A contain exactly 1 even digit which can be 2, 4, 6 or 8.

The other two digits are odd.

Their sums can be 1 + 3 = 4, 1 + 5 = 6, 1 + 7 = 8, 1 + 9 = 10,

3 + 5 = 8, 3 + 7 = 10, 3 + 9 = 12, 5 + 7 = 12,

5 + 9 = 14 or 7 + 9 = 16

If 'a' represents the even digit, then sum of all elements of A for a given value of 'a' will be

$222(a + 4) + 222(a + 6) + 222(a + 8) + 222(a + 8) + 222(a + 10) + 222(a + 12) + 222(a + 10) + 222\ (a + 12) + 222(a + 14) + 222(a + 16) = 222(10a + 100)$

But, a can be 2, 4, 6 or 8.

Sum of all elements of A is

$222(10 \times 2 + 100 + 10 \times 4 + 100 + 10 \times 6 + 100 + 10 \times 8 + 100)$

$= 222(400 + 200) = 133200.$

Alternatively,

The sum of digits in units place of all the numbers will be same as the sum in tens or hundreds place. So it

is enough to consider units place only. Now, position having even digit can be selected in 3 ways, the digit itself in 4 ways (2, 4, 6, 8) and other 2 odd digits in $5 \times 4 = 20$ ways. So in all there are $3 \times 4 \times 20 = 240$ numbers.

Number of numbers containing even digit in units place = $4 \times 20 = 80$.

So, other 160 numbers have digit 1, 3, 5, 7 or 9 in units place, with each digit appearing $\frac{160}{5} = 32$ times

So, sum in units place

$= 32(1 + 3 + 5 + 7 + 9) + 20(2 + 4 + 6 + 8)$

$= 32(5 \times 5) + 20(5 \times 4) = 1200$,

So, total = 1200 + 12000 + 120000 = 133200.

17. (a) : Note that it is a single - digit display in which some lines are permanently missing. So since the bottom most line is seen in the first display and it is working perfectly fine, it can't be missing in the last display *i.e.*, the last number does not have the bottom most line. Similarly, the second number does not have the top left line. Listing out all such possibilities, we see that the middle number must be 3 and last must be 4. The first number can be 0, 5, 6, 8 or 9. So the numbers could be 034, 534, 634, 834 or 934

18. (b): We need to find the number of points with integer coordinates inside the circle of radius 5 units and centre at (1, 2). The number of points will be the same for a circle at centre (0, 0) and radius of length 5 units. Then the equation of this circle is $x^2 + y^2 = 25$.

Consider the first quadrant.

The total number of integral points lying inside the circle but not on the axes in the first quadrant are the number of integer solutions of $x^2 + y^2 < 25$, $x \neq 0$, $y \neq 0$.

Thus, $(x, y) \equiv (1, 1), (1, 2), (1, 3), (1, 4), (2, 1), (2, 2), (2, 3), (2, 4), (3, 1), (3, 2) (3, 3), (4, 1), (4, 2)$.

$\therefore$ Number of such points = 13

$\therefore$ Number of such points in all the four quadrants = 52

Points on the axes are (0, 0), (0, 1), (0, 2), (0, 3), (0, 4), (0, –1), (0, –2), (0, –3), (0, –4), (1, 0), (2, 0), (3, 0), (4, 0), (–1, 0), (–2, 0), (–3, 0), (–4, 0)

$\therefore$ Number of such points = 17

Total number of required points = 52 + 17 = 69

19. (a) : Side of the second largest square is $\frac{1}{\sqrt{2}}$ times the radius of the largest quarter circle and so on and sides of square are $4, \frac{4}{\sqrt{2}}, 4\left(\frac{1}{\sqrt{2}}\right)^2, 4\left(\frac{1}{\sqrt{2}}\right)^3$ .....

$\therefore$ The required area

$$= \left[4^2 - \frac{\pi \times 4^2}{4}\right] + \left[(2\sqrt{2})^2 - \frac{\pi \times (2\sqrt{2})^2}{4}\right] + \left[2^2 - \frac{\pi \times 2^2}{4}\right] + \left[(\sqrt{2})^2 - \frac{\pi(\sqrt{2})^2}{4}\right]$$

$$= (16 + 8 + 4 + 2)\left(\frac{4 - \pi}{4}\right) = 6.42 \text{ cm}^2$$

20. (b): $\sqrt{1} = 1$, if $x = \sqrt{1}, y = 1$

$\sqrt{2} = 1.414$, if $x = \sqrt{2}, y = 1$

$\sqrt{3} = 1.732$, if $x = \sqrt{3}, y = 1$

$\sqrt{4} = 2$, if $x = \sqrt{4}, y = 2$

Thus, we can see that if p and $p + k$ are consecutive perfect squares, then value of y for $x = p$ to $p + k - 1$ are all equal to $\sqrt{p}$.

Using this logic, the required product

$$= \underbrace{1 \times 1 \times 1}_{\text{for } x=1 \text{ to } 3} \underbrace{2 \times 2 \times \ldots\ldots 2}_{\text{for } x=4 \text{ to } 8} \times \underbrace{3 \times 3 \times 3}_{\text{for } x=9 \text{ to } 15} \times \ldots\ldots \times \underbrace{9 \times 9 \times \ldots\ldots \times 9}_{\text{for } x=81 \text{ to } 99} \times \underbrace{10}_{\text{for } x=100}$$

$$1^3 \times 2^5 \times 3^7 \times 4^9 \times 5^{11} \times 6^{13} \times 7^{15} \times 8^{17} \times 9^{19} \times 10$$

$$= 2^5 \times 3^7 \times 2^{18} \times 5^{11} \times (2^{13} \times 3^{13}) \times 7^{15} \times 2^{51} \times 3^{38} \times 5^1 \times 2^1$$

$$= 2^{88} \times 3^{58} \times 5^{12} \times 7^{15}.$$

Note : It is sufficient to check the power of 7.

21. (b): The ant has to move 3 units to the north and 7 units to the west to reach point A. Since it moves 12 units in all and it does not move downwards, it makes 1 move to the east and one extra move to the west. The first two and the last two moves are as shown.

N W ..... N W

Now, we have to place 6 westward moves, 1 eastward move and 1 northward move. These can be placed in

${}^8C_6$ ways *i.e.*, $\frac{8!}{6! \times 2} \times 2 = 8 \times 7 = 56$

22. (c) :

	C_1	C_2	C_3	C_4
R_1				
R_2				

The first row can be filled with the four colours in 4! ways.

Once the first row is filled, the second row can be filled in D_4 ways, where D_4 is the number of dearrangements of 4.

$$D_4 = 4!\left[\frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!}\right] = 12 - 4 + 1 = 9$$

Total number of ways of filling the grid = 9 × 4!

23. (c) : $101^{76} = (100+1)^{76} = 100^{76} + 76\times100^{75} + \ldots + 76\times100^{1} + 1$

$= [100^{76} + 76\times100^{75} + \ldots + {}^{76}C_{74}\times100^{2}] + 7600 + 1$

$= M \times 100^2 + 7601$: M is some natural number.

$\therefore$ Third digit from the right in 101^{76} is 6.

24. (a) : $g^{22}(x) = x \quad \therefore \quad g^{22}(3) = 3$

$\therefore \; f(g^{22}(3)) = f(3) = \frac{1}{3}$

Now, $f(1) = f(2) + f(3)$. $\quad \therefore \quad f(2) = \frac{2}{3} - \frac{1}{3} = \frac{1}{3}$

Also, $f(0) = f(1) + f(2) \quad \therefore \quad f(0) = \frac{2}{3} + \frac{1}{3} = 1$

25. (c) : We have to calculate $f(7)$.

$f(3) = 1/3$, $f(2) = 1/3$

$\therefore \quad f(4) = f(2) - f(3) = \frac{1}{3} - \frac{1}{2} = 0$

Similarly, we have

$f(5) = 1/3$, $f(6) = -1/3$ and $f(7) = 2/3$

$\therefore \quad g^{2n+2}(x) = \frac{1}{f(7)} = \frac{3}{2}$

As $2n + 2$ is even for all values of n.

$$g^{2n+2}(x) = \frac{3}{2}$$

26. (a) : $\log_N 2M = 1 \Rightarrow 2M = N^1 \Rightarrow N = 2M$

Now, $\frac{1}{2}\log_M 0.25 + 2\log_5 N = \log_M N + 1$

i.e., $\frac{1}{2}\times\frac{\log 0.25}{\log M} + \frac{2\log N}{\log 5} = \frac{\log N}{\log M} + 1$

i.e., $\frac{1}{2}\times\frac{\log(0.5)^2}{\log M} + \frac{2\log 2M}{\log 5} = \frac{\log 2N}{\log M} + 1$

i.e., $\frac{1}{2}\times 2\times\frac{\log 1 - \log 2}{\log M} + \frac{\log 4M^2}{\log 5} = \frac{\log 2 + \log M}{\log M} + 1$

$\left(0.5 = \frac{1}{2} \text{ and } \log\frac{1}{2} = \log 1 - \log 2\right)$

i.e., $-\frac{\log 2}{\log M} + \frac{\log 4M^2}{\log 5} = 2 + \frac{\log 2}{\log M}$

i.e., $\frac{\log 4M^2}{\log 5} = 2 + \frac{\log 4}{\log M} \Rightarrow$ *i.e.*, $\frac{\log 4M^2}{\log 5} = \frac{\log 4M^2}{\log M}$

$\therefore \quad M = 5 \quad \Rightarrow \quad N = 10$

27. (a) : $2 - 0 = 2$ divides $p(2) - p(0) \Rightarrow p(2)$ is odd.

$p(0)$ is odd

Similarly 2 divides $p(4) - p(2) \Rightarrow p(4)$ is odd.

Thus, $p(k)$ is odd for all even integers k.

Also, 2 divides $p(3) - p(1) \Rightarrow p(3)$ is odd ($p(1)$ is odd)

Thus, as in previous case, $p(k)$ is odd for all odd integers k.

i.e., $p(k)$ is odd for all integers k.

i.e., $p(x) \neq 0$ for integer solution.

28. (d): Let m be any element in R. then $2n - m$, $4n - m$ and $6n - m$ cannot be the element in R.

Consider the following table.

m	$2n-m$	$4n-m$	$6n-m$
1	$2n-1$	$4n-1$	$6n-1$
2	$2n-2$	$4n-2$	$6n-2$
3	$2n-3$	$4n-3$	$6n-3$
...	...	...	...
$n-1$	$2n-(n-1)$ $= n+1$	$4n-(n-1)$ $= 3n+1$	$6n-(n-1)$ $= 5n+1$
n	$2n-n=n$	$4n-n=3n$	$6n-n=5n$

If the elements in the first column are the elements in R, then the elements in 2nd, 3rd and 4th column cannot be in R.

$\therefore$ Number of elements in $R = 5n - (n - 1 + n + 1 + 1)$

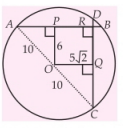
$= 3n - 1$

Alternatively, Check for different values of n

e.g., $n = 1, 2, 3, \ldots\ldots$

29. (a) : In ΔAPO, $l(AP) = \sqrt{10^2 - 6^2} = 8$ cm

$\Rightarrow \quad l(AB) = 16$ cm



Similarly in ΔCQO $l(CQ) = \sqrt{10^2 - (5\sqrt{2})^2}$

$= 5\sqrt{2}$ cm $\Rightarrow$ $l(CD) = 10\sqrt{2}$ cm

$\therefore \quad l(AR) = 8 + 5\sqrt{2}$, $l(RB) = 8 - 5\sqrt{2}$

$l(CR) = 6 + 5\sqrt{2}$ and $l(RD) = 5\sqrt{2} - 6$

$\therefore \quad l(AD) = \sqrt{l(AR)^2 + l(RD)^2}$

$= \sqrt{(8+5\sqrt{2})^2 + (5\sqrt{2}-6)^2} = \sqrt{200 + 20\sqrt{2}}$

$l(BC) = \sqrt{(8-5\sqrt{2})^2 + (6+5\sqrt{2})^2} = \sqrt{200 - 20\sqrt{2}}$

$l(AD) \times l(BC) = \sqrt{40000 - 800} = \sqrt{39200} = 140\sqrt{2}$ cm^2

Similarly, $l(AD) \times l(BC) = 20\sqrt{2}\ \text{cm}^2$ if we name point D as C and vice versa.

30. (d): If the polygon has n sides (n is even), $n/2$ of its diagonals will pass through the centre of the circle in which it is inscribed.
Also, the polygon has ${}^nC_2 - n$ diagonals in all.

$\therefore\ {}^nC_2 - n - \dfrac{n}{2} = 126$

$\therefore\quad n^2 - 4n - 252 = 0$

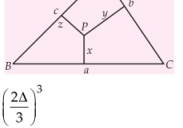
$\therefore\quad (n-18)(n+14) = 0\ .\quad \therefore\quad n = 18$

31. Let Δ be the area of the triangle ABC.
Δ = Area of ΔPBC + Area of ΔPCA + Area of ΔPAB

$$\Delta = \frac{1}{2}ax + \frac{1}{2}by + \frac{1}{2}cz \ \Rightarrow\ ax + by + cz = 2\Delta$$

Apply AM – GM inequality to ax, by, cz

$$\frac{ax+by+cz}{3} \geq (axbycz)^{1/3}$$



$$\Rightarrow\ axbycz \leq \left(\frac{2\Delta}{3}\right)^3$$

Since a, b, c are constants, xyz attains its maximum when $axbycz$ is maximum and it happens when $ax = by = cz = k$ (say)

$$\Rightarrow\ x = \frac{k}{a}, y = \frac{k}{b}, z = \frac{k}{c}\ ;\qquad x : y : z = \frac{1}{a} : \frac{1}{b} : \frac{1}{c}$$

$ax = by = cz$

$\Rightarrow$ Area of ΔPBC = Area of ΔPCA = Area of ΔPAB.

$\Rightarrow$ P is the centroid of the triangle ABC.

32. We have to find positive integers x, y, z such that $2^x + 2^y + 2^z = 2336$.
Let us first express 2336 in powers of 2 if possible.

$2336 = 2^5 \times 73$.

$\therefore\quad 2^z + 2^y + 2^z = 2^5 \times 73$.

$$\Rightarrow\ \frac{2^x}{2^5} + \frac{2^y}{2^5} + \frac{2^z}{2^5} = 73 \ \Rightarrow\ 2^{x-5} + 2^{y-5} + 2^{z-5} = 73$$

RHS is odd. LHS being powers of 2 is even. $\therefore$ One of the terms must be odd. This is possible only when 2^{x-5} or 2^{y-5} or 2^{z-5} is 1

Let $2^{x-5} = 1 \Rightarrow x - 5 = 0 \Rightarrow x = 5$

$\Rightarrow\quad 1 + 2^{y-5} + 2^{z-5} = 73 \ \Rightarrow\ 2^{y-5} + 2^{z-5} = 72$

$2^{y-5} + 2^{z-5} = 8 \times 9 = 2^3 \times 9$

$\Rightarrow\quad 2^{y-8} + 2^{z-8} = 9$

Again RHS is odd, LHS is even. Thus one of the two terms is odd.

Let $2^{y-8} - 1 \Rightarrow y - 8 = 0 \Rightarrow y = 8$

$\therefore\quad 2^{z-8} = 8 = 2^3$

$\Rightarrow\quad z - 8 = 3\ \Rightarrow\ z = 11$

Since the equation is symmetric in x, y, z, the solution set is $(x, y, z) = (5, 8, 11), (8, 5, 11), (11, 5, 8), (5, 11, 8), (8, 11, 5), (11, 8, 5)$

33. Consider $a^6 + 1$

$$a^6 + 1 = (a^2)^3 + 1 = (a^2+1)(a^4 - a^2 + 1)\ .$$

We are given the value of $a^5 - a^3 + a$ as 2.

$$a^6 + 1 = \left(\frac{a^2+1}{a}\right) a(a^4 - a^2 + 1)$$

$$= \left(\frac{a^2+1}{a}\right)(a^5 - a^3 + a) = \left(a + \frac{1}{a}\right)2$$

But $\dfrac{a + \frac{1}{a}}{2} \geq \sqrt{a.\frac{1}{a}} \ \Rightarrow\ a + \frac{1}{a} \geq 2$

The equality holds when $a = 1$ but $a = 1$ does not satisfy the equation.

$$a^5 - a^3 + a = 2 \qquad \therefore\quad a + \frac{1}{a} > 2$$

$\therefore\quad a^6 + 1 > 4\ \Rightarrow a^6 > 3\ \Rightarrow\ 3 < a^6$

To prove $a^6 < 4$.
From given, $a^5 - a^3 + a^2 = 2$ we have $a^3 + 2 = a^5 + a$.
Dividing throughout by a^3, we get

$$1 + \frac{2}{a^3} = a^2 + \frac{1}{a^2}$$

But $\dfrac{a^2 + \frac{1}{a^2}}{2} > \sqrt{a^2 \frac{1}{a^2}} \Rightarrow a^2 + \frac{1}{a^2} > 2$

$$\therefore\ 1 + \frac{2}{a^3} > 2 \ \Rightarrow\ \frac{2}{a^3} > 1 \ \Rightarrow\ a^3 < 2 \ \Rightarrow\ a^6 < 4$$

$\therefore\quad 3 < a^6 < 4$

34. Let α, β, γ be the roots of the given cubic $x^3 - ax + b = 0$, where $a > 0$ and $b > 0$. We have

$$\left.\begin{aligned} \alpha + \beta + \gamma &= 0 \\ \alpha\beta + \beta\gamma + \gamma\alpha &= -a \\ \alpha\beta\gamma &= -b \end{aligned}\right\} \qquad \ldots (1)$$

From the last of these equations, we see that either all the roots are negative or two are positive and

one is negative. However the second equation in (1) shows that all three cannot be negative. So two of α, β, γ are positive and the remaining root is negative. The first equation in (1) implies that the negative root is numerically larger than the other two positive roots. Hence we may assume that $\gamma < 0 < \alpha \le \beta$ where $|\alpha| \le |\beta| \le |\gamma|$.

We have

$$b - a\alpha = -\alpha\beta\gamma + \alpha(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= \alpha^2(\beta + \gamma) = -\alpha^3 < 0$$

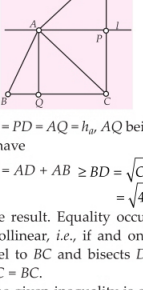
Since a is positive, we get $\frac{b}{a} < \alpha$ proving the first inequality.

Again, we have

$$3b - 2a\alpha = -3\alpha\beta\gamma + 2\alpha(\alpha\beta + \beta\gamma + \gamma\alpha)$$
$$= -\alpha\beta\gamma + 2\alpha^2\beta + 2\alpha^2\gamma = \alpha[2\alpha(\beta + \gamma) - \beta\gamma]$$
$$= \alpha[-2(\beta + \gamma)^2 - \beta\gamma) \quad (\text{since } \alpha = -(\beta + \gamma))$$
$$= -\alpha(2\beta + \gamma)(\beta + 2\gamma) = -\alpha(\beta - \alpha)(\gamma - \alpha)$$

Observe that $-\alpha < 0$, $\beta \ge \alpha$, $\gamma - \alpha < 0$. Hence $3b - 2a\alpha$ is non-negative. This proves the second inequality, $\alpha \le \frac{3b}{2a}$.

35. Draw a line l parallel to BC through A and reflect AC in this line to get AD. Let CD intersect l in P. Join BD



Observe that $CP = PD = AQ = h_a$, AQ being the altitude through A. We have

$$b + c = AC + AB = AD + AB \ge BD = \sqrt{CD^2 + CB^2}$$
$$= \sqrt{4h_a^2 + a^2}$$

which yields the result. Equality occurs if and only if B, A, D are collinear, *i.e.*, if and only if $AD = AB$ (as AP is parallel to BC and bisects DC) and this is equivalent to $AC = BC$.

Alternatively, the given inequality is equivalent to

$$(b + c)^2 - a^2 \ge 4h_a^2 = \frac{16\Delta^2}{a^2},$$

where Δ is the area of the triangle ABC. Using the identity

$$16\Delta^2 = [(b + c)^2 - a^2]\ a^2$$

we see that the inequality to be proved is $a^2 - (b - c)^2 \le a^2$ (here we use $a < b + c$) which is true. Observe that equality holds if and only if $b = c$.

36. If x, y, z exist, then some two of them share the same sign: say, x and y. Then $z = \frac{c}{\left(\frac{x}{y} + \frac{y}{x}\right)} > 0$. Thus

$a + b - c = \frac{2xy}{z}$, $b + c - a = \frac{2yz}{x}$, and $c + a - b = \frac{2zx}{y}$ are all positive, so a, b, c form a triangle.

Conversely, if there is a triangle with sides a, b, c, then let $u = b + c - a$, $v = c + a - b$, $w = a + b - c$; by the triangle inequality, these are all positive. If there did exist satisfactory x, y, z, then from above

$u = \frac{2yz}{x}$, $v = \frac{2zx}{y}$, $w = \frac{2xy}{z}$.

Solving these equations gives

$$x = \sqrt{\frac{vw}{2}},\ y = \sqrt{\frac{wu}{2}},\ z = \sqrt{\frac{uv}{2}}$$

and these values indeed satisfy the equations.

37. Let integer $n = p_1^{a_1} p_2^{a_2} \cdots p_m^{a_m}$ with $p_1, ..., p_m$ distinct primes. Then n has $(a_1 + 1)(a_2 + 1) \ldots (a_n + 1)$ divisors. Since $18 = 2 \cdot 3^2$, it has 6 factors: 1, 2, 3, 6, 9, 18.

Since d has 16 divisors, we know that $d = 2 \cdot 3^3 \cdot p$ or $d = 2 \cdot 3^7$. If $b = 2 \cdot 3^7$, $d_8 = 54$, $d_9 = 81$ and $d_9 - d_8 \ne 17$. Thus $d = 2 \cdot 3^3 \cdot p$ for some prime $p > 18$. If $p < 27$, then $d_7 = p$, $d_8 = 27$, $d_9 = 2p = 27 + 17 = 44$ $\Rightarrow$ $p = 22$, a contradiction. Thus $p > 27$. If $p < 54$, $d_7 = 27$, $d_8 = p$, $d_9 = 54 = d_8 + 17 \Rightarrow p = 37$. If $p > 54$, then $d_7 = 27$, $d_8 = 54$, $d_9 = d_8 + 17 = 71$. We obtain two solutions for the problem: $2 \cdot 3^3 \cdot 37 = 1998$ and $2 \cdot 3^3 \cdot 71 = 3834$.

38. **1st solution:** Let $a = BC$, $b = CA$, $c = AB$, $\alpha = \angle CAB$, $\beta = \angle ABC$, $\gamma = \angle BCA$,

and let R and r be the circumradius and inradius of ABC, respectively. Applying law of sines to ABC, we have $a = 2R\sin\alpha$, $b = 2R\sin\beta$, $c = 2R\sin\gamma$

Since $\beta = 45°$, $\sin\beta = \frac{\sqrt{2}}{2}$, $\tan\left(\frac{\beta}{2}\right) = (\sqrt{2} - 1)$, and

$$\sin\gamma = \sin(135° - \alpha) = \frac{\sqrt{2}(\sin\alpha + \cos\alpha)}{2} \quad \ldots (1)$$

Thus,

$$r = \frac{(c + a - b)}{2}\tan\left(\frac{\beta}{2}\right) = R(\sqrt{2} - 1)(\sin\alpha + \sin\gamma - \sin\beta).$$

From Euler's formula $OI^2 = R(R - 2r)$, we have

$$OI^2 = R^2(1 - 2(\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2} - 1)) \quad \ldots (2)$$

Since $\sqrt{2}OI = AB - AC$.

$OI^2 = (c - 2(\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2} - 1))$ ... (3)

From (1) and (2), we obtain

$2(\sin\gamma - \sin\beta)^2 = (1 - 2(\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2} - 1))$

$\Leftrightarrow 1 - 2(\sin\gamma - \sin\beta)^2 = (\sin\alpha + \sin\gamma - \sin\beta)(\sqrt{2} - 1)$

$\Leftrightarrow 1 - 2\sin^2\gamma + 2\sqrt{2}\sin\gamma - 1 = 2(\sin\alpha + \sin\gamma)(\sqrt{2} - 1) - (2 - \sqrt{2})$

$\Leftrightarrow -(\sin\alpha + \cos\alpha)^2 + 2(\sin\alpha + \cos\alpha)$

$= (2\sqrt{2} - 2)\sin\alpha + (2 - \sqrt{2})(\sin\alpha + \cos\alpha) - (2 - \sqrt{2})$

$\Leftrightarrow -1 - 2\sin\alpha\cos\alpha = (\sqrt{2} - 2)\sin\alpha - \sqrt{2}\cos\alpha - (2 - \sqrt{2})$

$\Leftrightarrow 2\sin\alpha\cos\alpha - (2 - \sqrt{2})\sin\alpha - \sqrt{2}\cos\alpha + (\sqrt{2} - 1) = 0$

$\Leftrightarrow (\sqrt{2}\sin\alpha - 1)(\sqrt{2}\cos\alpha - \sqrt{2} + 1) = 0$

Thus $\sin\alpha = \dfrac{\sqrt{2}}{2}$ or $\alpha = 1 - \dfrac{\sqrt{2}}{2}$,

$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \dfrac{\sqrt{4\sqrt{2} - 2}}{2}$

2nd solution: Let I_c, I_a, I_b be the feet of perpendiculars from I to AB, BC, CA, respectively. Let D be the foot of the perpendicular from O to BC. Thus OD is the perpendicular bisector of BC and $BD = CD$. From equal tangents, we have $AI_c = AI_b$, $BI_a = BI_c$, $CI_a = CI_b$.

We have $\sqrt{2}OI = c - b$

$= (AI_c + I_cB) - (AI_b + I_bC) = I_cB - I_bC = BI_a - I_aC$

Since $c > b$, D is on BI_a. We have $BI_a = BD + DI_a$, $I_aC = CD - DI_a$. So $\sqrt{2}OI = 2DI_a$, *i.e.*, $OI = \sqrt{2}DI_a$. Thus line OI and line DI_a form a 45° angle, which implies that either $OI \perp AB$ or $OI \parallel AB$.

(a) $OI \perp AB$. Then OI is the perpendicular bisector of AB. Thus $AC = BC$, $\alpha = \beta = 45°$, and $\sin\alpha = \dfrac{\sqrt{2}}{2}$.

(b) $OI \parallel AB$. Let E be the foot of the perpendicular from O to AB. So $\angle AOE = \angle C = \gamma$

$R\cos\angle AOE = R\cos\gamma = OE = II_c = r$,

Since $r = 4R\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$, we have

$\cos\gamma = 4\sin\dfrac{\alpha}{2}\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2} = 2\sin\dfrac{\beta}{2}\left(2\sin\dfrac{\alpha}{2}\sin\dfrac{\gamma}{2}\right)$

$= 2\sin\dfrac{\beta}{2}\left(-\cos\left(\dfrac{\alpha+\gamma}{2}\right) + \cos\left(\dfrac{\alpha-\gamma}{2}\right)\right)$

$= 2\sin\dfrac{\beta}{2}\left(-\sin\dfrac{\beta}{2} + \cos\left(\dfrac{\alpha-\gamma}{2}\right)\right)$

$= -2\sin^2\dfrac{\beta}{2} + 2\sin\dfrac{\beta}{2}\cos\left(\dfrac{\alpha-\gamma}{2}\right)$

$= \cos\beta - 1 + \sin\left(\dfrac{\alpha+\beta-\gamma}{2}\right) + \sin\left(\dfrac{\beta+\gamma-\alpha}{2}\right)$

$= \cos\beta - 1 + \sin(90° - \gamma) + \sin(90° - \alpha)$

$= \cos\beta - 1 + \cos\gamma + \cos\alpha$,

which implies that $\cos\alpha = 1 - \cos\beta = 1 - \dfrac{\sqrt{2}}{2}$ and

$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \dfrac{\sqrt{4\sqrt{2} - 2}}{2}$.

39. 1st solution: Let $BC = a$, $CA = b$, $AB = c$. We have $A = 2B$ and $C = 180° - 3B$. By the law of sines,

$$\frac{b}{\sin B} = \frac{a}{\sin A} = \frac{c}{\sin C}.$$

Since $\sin A = \sin 2B = 2\sin B\cos B\sin C = \sin 3B$

$= 3\sin B - 4\sin^3 B$, we have

$a = 2b\cos B$, $c = b(3 - 4\sin^2 B) = b(4\cos^2 B - 1)$

and hence $a^2 = b(b + c)$. Since we are looking for a triangle of smallest perimeter, we may assume that $gcd(a, b, c) = 1$. In fact, $gcd(b, c) = 1$, since any common factor of b and c would be a factor of a as well. We notice that since a perfect square a^2 is being expressed as the product of two relatively prime integers b and c, it must be the case that both b and $b + c$ are perfect squares. Thus, for some integers m and n, with $gcd(m, n) = 1$, we have $b = m^2$, $b + c = n^2$, $a = mn$, $2\cos B = \dfrac{n}{m} = \dfrac{a}{b}$. Since $C > 90°$, we have $0 < B < 30°$ and $\sqrt{3} < 2\cos B = \dfrac{n}{m} < 2$

It is easy to check that $(m, n) = (4, 7)$ is the smallest pair that generates a triangle $(a, b, c) = (28, 16, 33)$ that meets all the conditions.

2nd solution: We use same notations as those in the first solution. Let the angle bisector of $\angle CAB$ meet BC at D. Since $\angle BAD = \angle ABD$, we let $AD = BD = x$. We have $\angle ACD = \angle B$, $\angle ACB = \angle ACD$, so triangles ABC and DAC are similar. We have

$$\frac{x}{c} = \frac{b}{a} = \frac{a - x}{b}$$

which leads to $ax = bc$, $b^2 = a^2 - ax \Rightarrow a^2 = b(b + c)$, and the rest is the same.

40. 1st solution: Let p be a prime and 'a' be a positive integer. Let $\left(\dfrac{a}{p}\right)$ be the Legendre symbol, *i.e.*, $\left(\dfrac{a}{p}\right)$ will have value 1 if a is a quadratic residue module

p, –1 if a is a quadratic non-residue modulo p, and 0 if $\frac{p}{a}$. For odd primes p and q, we have *LQR* (Law of Quadratic Reciprocity):

$$\left(-\frac{1}{p}\right)=(-1)^{\frac{(p-1)}{2}},\ \left(\frac{p}{q}\right)\left(\frac{q}{p}\right)=(-1)^{\frac{(p-1)(q-1)}{4}}$$

If m is odd, then $(m+3)^n+1$ is odd and A is odd. Now we suppose that m is even. Since A is an integer,

$$0 \equiv (m+3)^n + 1 = m^n + 1 (\text{mod } 3)$$

so $n = 2k + 1$ is odd and $m \equiv -1$ (mod 3). We consider the following cases :

(a) $m = 8m'$ for some positive integer m'. Then

$$(m+3)^n + 1 \equiv 3^{2k+1} + 1 \equiv 4 \pmod 8$$

and $3m \equiv 0$ (mod 8). So A is not an integer.

(b) $m = 2m'$ for some odd positive integer m', *i.e.*, $m \equiv 2$ (mod 4).

Then, $(m+3)^n + 1 \equiv (2+3)^n + 1 \equiv 2 (\text{mod } 4)$

and $3m \equiv 2$ (mod 4). So A is odd.

(c) $m = 4m'$ for some odd positive integer m'. Since $m \equiv -1$ (mod 3), there exists an odd prime p such that $p \equiv -1$ (mod 3) and $p \mid m$. Since A is an integer,

$$0 \equiv (m+3)^n + 1 \equiv 3^{2k+1} = 1 (\text{mod } m)$$

and $3^{2k+1} = -1 (\text{mod } p)$. Let a be a primitive root module p; let b be a positive integer such that $3 \equiv a^b \pmod p$.

Thus $a^{(2k+1)b} \equiv -1 (\text{mod } p)$. Note that $\left(\frac{p}{3}\right)=\left(\frac{-1}{3}\right)=-1$. We consider the following cases :

(i) $p \equiv 1$ (mod 4). From *LQR*, $\left(\frac{-1}{p}\right)=1$, so

$$a^{2c} \equiv -1 \equiv a^{(2k+1)b} (\text{mod } p)$$

for some positive integer c. Therefore be is even and $\left(\frac{3}{p}\right)=1$. Again, from *LQR*, we have

$$-1=\left(\frac{3}{p}\right)\left(\frac{p}{3}\right)=(-1)^{\frac{(3-1)(p-1)}{4}}=1,$$ a contradiction.

(ii) $p \equiv 3 (\text{mod } 4)$. From *LQR*, $\left(\frac{-1}{p}\right)=-1$, so

$$a^{2c+1} \equiv -1 \equiv a^{(2k+1)b} (\text{mod } p)$$

For some positive integer c. Therefore be is even and $\left(\frac{3}{p}\right)=-1$. Again, from *LQR*, we have

$$1=\left(\frac{3}{p}\right)\left(\frac{p}{3}\right)=(-1)^{\frac{(3-1)(p-1)}{4}}=-1,$$

a contradiction.

Thus for $m = 4m'$ and m' is odd, A is not an integer. From the above, we see that if A is an integer, A is odd.

2nd solution : We prove by contradiction. Assume, on the contrary, that A is even. Then m is even. Since A is an integer, $0 \equiv (m+3)^n + 1 \equiv m^n + 1 (\text{mod } 3)$ yields $n = 2k + 1$ and $m = 3t + 2$. Let $m = 2\,m_1$, where $l \geq 1$ and m_1 is odd. In fact $l > 1$, as otherwise $m \equiv 2 (\text{mod } 4)$,

$$(m+3)^n + 1 \equiv (2+3)^n + 1 \equiv 2 \pmod 4$$

and $A=\frac{(m+3)^n+1}{2m'}$ is odd.

Since A is an integer, we have

$$0 \equiv (m+3)^n + 1 \equiv 3^{2k+1} + 1 (\text{mod } m) \qquad ...(2)$$

From (2), we have $2^l \mid (3^{2k+1}+1)$.

But $3^{2k+1} + 1 = 9^k \times 1 \equiv 4 (\text{mod } 8)$,

so $l = 2$, $m = 4m_1$, and $m_1 \equiv m = 2$ (mod 3),

From (2), we also have $m_1 \mid (3^{2k+1}+1)$, which implies that $m_1 \mid a^2 + 3$, where $a = 3^k + 1$. Since

$$gcd(m_1, 2) = gcd(m_1, 3) = 1$$

and $m_1 \equiv 2$ (mod 3), $m_1 = 6s + 5$.

Since $gcd(m_1, a) = gcd(m_1, 3) = 1$. Thue's lemma implies that there exist integers x and y such that $1 \leq x, |y| \leq \left\lfloor \sqrt{m_1} \right\rfloor$ and $m_1 \mid ax + y$. Since $m_1 \equiv 2$ (mod 3), m_1 is not a perfect square and $1 \leq x$, $|y| < \sqrt{m_1}$. Now

$$m_1 \mid a^2 + 3 \Leftrightarrow a^2 + 3 \equiv 0 \pmod{m_1}$$

$$m_1 \mid ax + y \Leftrightarrow ax \equiv -y \pmod{m_1}$$

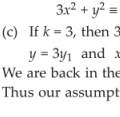
imply that $3x^2 + y^2 \equiv 0 \pmod{m_1}$, *i.e.*, $3x^2 + y^2 = km_1$. But $3x^2 + y^2 < 4m_1$ gives $k = 1, 2, 3$.

(a) If $k = 1$, then $3x^2 + y^2 = m_1 = 6s + 5$ yields $y^2 \equiv 2 (\text{mod } 3)$, which is impossible.

(b) If $k = 2$, then $3x^2 + y^2 = 2m_1 = 12s + 10$ yields $3x^2 + y^2 \equiv 2 (\text{mod } 4)$, which is impossible.

(c) If $k = 3$, then $3x^2 + y^2 = 3m_1$ yields $y = 3y_1$ and $x^2 + 3y_1^2 = m_1$.

We are back in the first case, which is impossible. Thus our assumption is wrong and A is odd. ■■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MOCK TEST PAPER 2014

JEE Advanced

Exam on 25th May 2014

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SECTION-I

Single Correct Answer Type

This section contains multiple choice questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If $9 + f''(x) + f'(x) = x^2 + f^2(x)$ be the differential equation of a curve and let P be the point of minima of this curve then the number of tangents which can be drawn from P to the circle $x^2 + y^2 = 8$ is

(a) 2 (b) 1
(c) 0 (d) either 1 or 2

2. Number of integral values of k for which exactly three roots of the equation
$\sin^2 x + (\cos x - 1)\sin x - \cos x - k \sin x + k = 0$,
$x \in (0, 2\pi)$ are real, is

(a) 0 (b) 3 (c) 4 (d) 2

3. If a right angled triangle is inscribed in a rectangular hyperbola $xy = 4$, then the angle between the tangent at the right angled vertex and the hypotenuse of the triangle is

(a) 45° (b) 30° (c) 60° (d) 90°

4. A function $f : R \to R$ satisfies the equation $f(x) f(y) - f(xy) = x + y \ \forall \ x, y \in R$ and $f(1) > 0$, also $h(x) = f(x) f^{-1}(x)$, then length of longest interval in which $h(\sin x + \cos x)$ is strictly increasing is

(a) $\pi/4$ (b) $\pi/3$ (c) $\pi/2$ (d) π

5. $\begin{vmatrix} \sin^2 x & \cos^2 x & \alpha \\ \cos^2 x & \alpha & \sin^2 x \\ \alpha & \sin^2 x & \cos^2 x \end{vmatrix} = 0$, where α is a particular positive value and $x \in \left(0, \frac{\pi}{2}\right)$, then x must be

(a) $\pi/6$ (b) $\pi/4$ (c) $\pi/3$ (d) $\pi/12$

6. The perimeter of a parallelogram is 40. All the sides of the parallelogram are natural numbers, and at least one of its diagonals is a natural number. The number of non congruent parallelograms so formed is

(a) 10 (b) 30 (c) 60 (d) 100

7. The number of points in the complex plane that satisfy the conditions $|z - 2| = 2$, $z(1 - i) + \bar{z}(1 + i) = 4$ (where $i = \sqrt{-1}$) is

(a) 0 (b) 1
(c) 2 (d) more than two

8. If $P = \begin{bmatrix} \frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $Q = PAP^T$, then $P^T(Q^{2005})P$ is equal to

(a) $\begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$ (b) $\begin{bmatrix} \frac{\sqrt{3}}{2} & 2005 \\ 1 & 0 \end{bmatrix}$

(c) $\begin{bmatrix} 1 & 2005 \\ \frac{\sqrt{3}}{2} & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 1 & \frac{\sqrt{3}}{2} \\ 0 & 2005 \end{bmatrix}$

9. The number of ways in which 30 coins of one rupee each be given to six persons, so that none of them receives less than 4 rupees is

(a) 231 (b) 462 (c) 693 (d) 924

10. Let $f(x) = \begin{cases} \frac{a|x^2 - x - 2|}{2 + x - x^2}, & x < 2 \\ b, & x = 2 \\ \frac{x - [x]}{x - 2}, & x > 2 \end{cases}$

([.] denotes the greatest integer function)
If $f(x)$ is continuous at $x = 2$, then

(a) $a = 1, b = 2$ (b) $a = 1, b = 1$
(c) $a = 0, b = 1$ (d) $a = 2, b = 1$

11. If $2f(x - 1) - f\left(\frac{1 - x}{x}\right) = x$, then $f(x)$ is

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(a) $\frac{1}{3}\left\{2(1+x)+\frac{1}{(1+x)}\right\}$

(b) $2(x-1)-\frac{(1-x)}{x}$ (c) $x^2+\frac{1}{x^2}+4$

(d) $\frac{1}{4}\left\{(x+2)+\frac{1}{(x+2)}\right\}$

12. The remainder $R(x)$ obtained by dividing the polynomial x^{100} by the polynomial x^2-3x+2 is

(a) $2^{100}-1$ (b) $(2^{100}-1)x-2(2^{99}-1)$

(c) $2^{100}x-3\cdot2^{100}$ (d) $(2^{100}-1)x+2(2^{99}-1)$

13. The smallest positive root of the equation $\tan x - x = 0$ lies in

(a) $\left(0,\frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{2},\pi\right)$

(c) $\left(\pi,\frac{3\pi}{2}\right)$ (d) $\left(\frac{3\pi}{2},2\pi\right)$

14. Let $\vec{v}=2\hat{i}+\hat{j}-\hat{k}$ and $\vec{w}=\hat{i}+3\hat{k}$. If $\vec{u}$ is a unit vector, then maximum value of the scalar triple product $[\vec{u}\ \vec{v}\ \vec{w}]$ is

(a) -1 (b) $\sqrt{10}+\sqrt{6}$

(c) $\sqrt{59}$ (d) $\sqrt{60}$

15. If $y = f(x)$ passing through (1, 2) satisfies the differential equation $y(1+xy)dx - xdy = 0$, then

(a) $f(x)=\frac{2x}{2-x^2}$ (b) $f(x)=\frac{x+1}{x^2+1}$

(c) $f(x)=\frac{x-1}{4-x^2}$ (d) $f(x)=\frac{4x}{1-2x^2}$

16. Let $f(x)=\begin{cases}x+1, x<0\\ |x-1|, x\geq 0\end{cases}$ and $g(x)=\begin{cases}x+1, & x<0\\ (x-1)^2, & x\geq 0\end{cases}$ then the number of points at which the function $g(f(x))$ is not differentiable is

(a) 0 (b) 1 (c) 2 (d) 3

17. Number of points having position vector $a\hat{i}+b\hat{j}+c\hat{k}$, where $a, b, c \in \{1, 2, 3, 4, 5\}$ such that $2^a+3^b+5^c$ is divisible by 4 is

(a) 140 (b) 70 (c) 100 (d) 120

18. If $\sum_{i=1}^{n}\vec{a}_i=\vec{0}$ when $|\vec{a}_i|=1\ \forall\ i$, then the value of $\sum_{1\leq i<j\leq n}\sum\vec{a}_i\cdot\vec{a}_j$ is

(a) $-n/2$ (b) $1-n$ (c) $n/2$ (d) n

19. Area of the trapezium whose vertices lies on the parabola $y^2 = 4x$ and its diagonals pass through (1, 0) and having length 25/4 unit each, is

(a) $\frac{75}{4}$ sq.units (b) $\frac{625}{16}$ sq.units

(c) $\frac{25}{4}$ sq.units (d) $\frac{25}{8}$ sq.units

20. If $\arg\left(\frac{z-1}{z+1}\right)=\pm\frac{\pi}{4}$ then length of the path traced by the points in the locus is

(a) $3\sqrt{2}\pi$ (b) 3π (c) $\frac{\pi}{\sqrt{2}}$ (d) $\sqrt{2}\pi$

21. An unbiased die with faces numbered 1, 2, 3, 4, 5, 6 is thrown n times and the list of n numbers shown up is noted. Then the probability that among the numbers 1, 2, 3, 4, 5, 6 only three numbers appear in this list is

(a) $\frac{3^n\cdot{}^6C_3}{6^n}$ (b) $\frac{(3^n-3\cdot2^n+3)\,{}^6C_3}{6^n}$

(c) $\frac{(3^n-3)\,{}^6C_3}{6^n}$ (d) $\frac{3^n}{6^n}$

22. $\int\frac{\cos 9x+\cos 6x}{2\cos 5x-1}dx = A\sin 4x + B\sin x + C$, then $A + B =$

(a) 1/2 (b) 3/4 (c) 5/4 (d) 7/4

23. $\lim_{x\to0}\frac{(1+x)^{1/x}-e}{x} =$

(a) 0 (b) e (c) $e/2$ (d) $-e/2$

24. The number of irrational terms in the expansion of $(2^{1/4}+3^{1/10})^{55}$ is

(a) 47 (b) 56 (c) 50 (d) 6

25. Let $f(x)=\frac{4^x}{4^x+2}$. The sum of the series

$$f\left(\frac{1}{2008}\right)+f\left(\frac{2}{2008}\right)+f\left(\frac{3}{2008}\right)+....+f\left(\frac{2007}{2008}\right)=$$

(a) 1004 (b) 1004.5

(c) 1003.5 (d) 1003

26. The sum of the infinite series $\cot^{-1}2+\cot^{-1}8+\cot^{-1}18+\cot^{-1}32+....$ to infinite terms is

(a) π (b) $\pi/2$

(c) $\pi/4$ (d) none of these

27. If D is the midpoint of side BC of a ΔABC and AD is perpendicular to BC then

(a) $3a^2=b^2-3c^2$ (b) $3b^2=a^2-c^2$

(c) $b^2=a^2-c^2$ (d) $a^2+b^2=5c^2$

28. In a certain test there are n questions. In this test 2^{n-i} students gave wrong answers to atleast i questions, where $i = 1, 2, 3, \ldots, n$. If the total number of wrong answers given is 2047. Then n is
(a) 10 (b) 11 (c) 12 (d) 13

29. The domain of $f(x) = \dfrac{1}{\sqrt{x^{12} - x^9 + x^4 - x + 1}}$ and range of $g(x) = \dfrac{x^3 - 8}{x - 2}$ respectively
(a) $(-\infty, \infty), [3, \infty)$ (b) $(0, \infty), [3, \infty)$
(c) $(-\infty, \infty)$; $[3, 12) \cup (12, \infty)$
(d) $(0, \infty)$; $[3, 12) \cup (12, \infty)$

30. If a complex number z lies on a circle of radius 1/2 then the complex number $-1 + 4z$ lies on a circle of radius
(a) 1/2 (b) 1 (c) 2 (d) 4

SECTION-II

Multiple Correct Answer Type

This section contains multiple correct answer(s) type questions. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

31. If $\vec{a}, \vec{b}, \vec{c}$ are vectors forming right hand triad. Let $\vec{p} = \dfrac{\vec{b} \times \vec{c}}{[\vec{a}, \vec{b}, \vec{c}]}, \vec{q} = \dfrac{\vec{c} \times \vec{a}}{[\vec{a}, \vec{b}, \vec{c}]}, \vec{r} = \dfrac{\vec{a} \times \vec{b}}{[\vec{a}, \vec{b}, \vec{c}]}, x \in R_+$ then
(a) $x[\vec{a}, \vec{b}, \vec{c}] + \dfrac{[\vec{p}, \vec{q}, \vec{r}]}{x}$ has least value 2
(b) $x^4[\vec{a}, \vec{b}, \vec{c}]^2 + \dfrac{[\vec{p}, \vec{q}, \vec{r}]}{x^2}$ has least value $\left(\dfrac{3}{2^{2/3}}\right)$
(c) $[\vec{p}, \vec{q}, \vec{r}] > 0$ (d) none of these

32. If the cubic equation $4x^3 - 3x + r = 0$ has exactly one root in $(-1, 0)$, then r cannot be
(a) $-\dfrac{2}{3}$ (b) $\dfrac{2}{3}$ (c) $\dfrac{\sqrt{3}}{2}$ (d) $-\dfrac{\sqrt{3}}{2}$

33. Which of the following statements are true with respect to definite integrals ?
(a) $\int_0^{\pi/2} \cos^m x \sin^m x \, dx = 2^{-m} \int_0^{\pi/2} (\cos x)^m \, dx, m \in N$
(b) $\int_0^{\pi} \dfrac{x \sin x}{1 + \cos^2 x} dx = \dfrac{\pi}{2} \int_0^1 \dfrac{dx}{1 + x^2}$
(c) $\int_0^{\pi/2} \dfrac{\cos 3x + 1}{2 \cos x - 1} dx = \int_0^{\pi/2} \dfrac{1 - \cos 3x}{1 + 2 \cos x} dx$
(d) $\int_0^{\pi/2} \left(\dfrac{1 + \sin 3x}{1 + 2 \sin x}\right) dx = \int_0^{\pi/2} \dfrac{\cos 3x + 1}{2 \cos x - 1} dx$

34. Suppose that a, b, c are real numbers such that $a + b + c = 1$. If the matrix $A = \begin{bmatrix} a & b & c \\ b & c & a \\ c & a & b \end{bmatrix}$ be an orthogonal matrix, then
(a) A is an involutory matrix
(b) $|A|$ is negative
(c) $a^3 + b^3 + c^3 - 3abc = 1$
(d) atleast one of a, b, c is negative

35. Given $f(x)$ is an odd function periodic with period 2, continuous $\forall$ real x and $g(x) = \int_0^x f(t)dt$ then
(a) $g(x)$ is even function
(b) $g(2n) = 0$ for $n \in z^+$
(c) $g(x)$ is periodic with period 2
(d) $g(2n) = 1$ for $n \in z^+$

36. If $a < 0$ then the value of x satisfying $x^2 - 2a|x - a| - 3a^2 = 0$
(a) $x = a(1 - \sqrt{2})$ (b) $x = a(1 + \sqrt{2})$
(c) $x = a(-1 + \sqrt{6})$ (d) $x = a(-1 - \sqrt{6})$

37. If l is the length of the median from the vertex A to the side BC of ΔABC then
(a) $4l^2 = a^2 + 4bc \cos A$ (b) $4l^2 = 2b^2 + 2c^2 - a^2$
(c) $4l^2 = b^2 + c^2 + 2bc \cos A$
(d) $4l^2 = (2s - a)^3 - 4bc \sin^2(A/2)$

38. A tangent to the ellipse $\dfrac{x^2}{9} + \dfrac{y^2}{4} = 1$ is cut by the tangents at the extremities of the major axes at T & T' and the circle on TT' as diameter passes through the point Q then Q may be
(a) $(-\sqrt{5}, 0)$ (b) $(\sqrt{5}, 0)$
(c) $(3, 0)$ (d) $(0, 0)$

39. The equation of a conic is $y^2 + 2ax + 2by + c = 0$, then
(a) It is an ellipse (b) it is a parabola
(c) Its latus-rectum = a (d) Its latus rectum = $2a$

40. If the tangent at the point $\left(4\cos\theta, \dfrac{16}{\sqrt{11}}\sin\theta\right)$ to the ellipse $16x^2 + 11y^2 = 256$ is also a tangent to the circle $x^2 + y^2 - 2x = 15$, then θ is
(a) $\pi/3$ (b) $2\pi/3$ (c) $-\pi/3$ (d) $5\pi/3$

41. If P_1, P_2, P_3 denote the distances of the plane $2x - 3y + 4z + 2 = 0$ from the planes $2x - 3y + 4z + 6 = 0$, $4x - 6y + 8z + 3 = 0$ and $2x - 3y + 4z - 6 = 0$ respectively, then
(a) $P_1 + 8P_2 - P_3 = 0$ (b) $P_3 = 16P_2$
(c) $8P_2 = P_1$ (d) $P_1 + 2P_2 + 3P_3 = \sqrt{29}$

42. In a ΔABC, $2\cos\dfrac{(A-C)}{2} = \dfrac{a+c}{\sqrt{a^2+c^2-ac}}$ then

(a) $B = \pi/3$ (b) $B = C$

(c) A, B, C are in A.P. (d) $B + C = A$

SECTION-III

Paragraph Type

This section contains paragraphs. Based upon each paragraph, multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

Paragraph for Questions Nos. 43 to 45

If $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ represents an ellipse, then $h^2 < ab$ and $abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$. If for every point (x_1, y_1) satisfying above equation $(2h - x_1, 2k - y_1)$ also satisfy it, then (h, k) is centre of it. The lengths of semi major axis and minor axis are nothing but the maximum and minimum values of the distance of a point lying on the curve from its centre.

43. For the ellipse $2x^2 - 2xy + 4y^2 - (3+\sqrt{2}) = 0$, the inclination of the major axis of it with x-axis is

(a) $\pi/12$ (b) $\pi/8$ (c) $3\pi/8$ (d) $5\pi/8$

44. If the given ellipse is rotated so that it has its major axis coincident with x-axis, then its equation becomes

(a) $3(x^2 + y^2 - 1) = (y^2 - x^2 + 1)\sqrt{2}$

(b) $3(x^2 + y^2 - 1) = (y^2 - x^2 - 1)\sqrt{2}$

(c) $3(x^2 - y^2 + 1) = (y^2 - x^2 - 1)\sqrt{2}$

(d) $\sqrt{2}(x^2 - y^2 + 1) = 3(x^2 + y^2 - 1)$

45. The product of perpendiculars from the foci to any tangent to above given ellipse is

(a) 4 (b) 2 (c) 1 (d) 1/2

Paragraph for Questions Nos. 46 to 48

Suppose equation is $f(x) - g(x) = 0$ or $f(x) = g(x) = y$ say then draw the graphs of $y = f(x)$ and $y = g(x)$. If graphs of $y = f(x)$ and $y = g(x)$ cuts at one, two, three, ... no points, then the number of solutions are one, two, three, ..., zero respectively.

46. The number of solutions of $\sin x = \dfrac{|x|}{10}$ is

(a) 4 (b) 6

(c) 18 (d) None of these

47. Total number of solutions of $\sin\{x\} = \cos\{x\}$, when $\{.\}$ denotes the fractional part, in $[0, 2\pi]$ is equal to

(a) 3 (b) 6

(c) 7 (d) None of these

48. Total number of solutions of $\cos 2x = |\sin x|$ where $x \in \left(-\dfrac{\pi}{2}, \pi\right)$

(a) 3 (b) 6 (c) 5 (d) 4

Paragraph for Question Nos.49 to 51

A and B are two matrices of same order 3×3 where

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 5 & 6 & 8 \end{pmatrix}, B = \begin{pmatrix} 3 & 2 & 5 \\ 2 & 3 & 8 \\ 7 & 2 & 9 \end{pmatrix}$$

49. The value of $|\text{adj}(\text{adj}A)|$ is equal to

(a) 9 (b) 16 (c) 25 (d) –1

50. The value of $|\text{adj}(AB)|$ is equal to

(a) 24 (b) 24^2 (c) 24^3 (d) 65

51. The value of $|\text{adj}(\text{adj}(\text{adj}(\text{adj}A)))|$ is equal to

(a) 2^4 (b) 2^9 (c) –1 (d) 2^{19}

SECTION-IV

Matrix Match Type

This section contains 6 questions. Each question contains statements given in two columns which have to be matched. Statements in Column I are labeled as A, B, C and D whereas statements in Column II are labeled as p, q, r and s. The answers to these questions have to be appropriately bubbled as illustrate din the following examples. If the correct matches are A – p, s ; B – q, r ; C – p, q ; D – s, then the correctly bubbled matrix will look like the following:

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

52. Match the extrema of the functions:

	Column-I		Column-II
(A)	$y = x + \sqrt{1-x}$	(p)	$y_{\min} = 0$
(B)	$y = x^2 e^{-x}$	(q)	$y_{\min} = 1$
(C)	$y = x\sqrt{2-x^2}$	(r)	$y_{\max} = 5/4$
(D)	$y = \dfrac{e^{ax} + e^{-ax}}{2}$	(s)	$y_{\min} = -1$

53. Match the following:

	Column-I		Column-II
(A)	The distance between circumcentre and incentre of a triangle, the affixes of whose vertices are 1, ω, ω^2 is ($1, \omega, \omega^2$ are cube roots of unity)	(p)	6

(B)	If ${}^{2n}C_4$, ${}^{2n}C_5$, ${}^{2n}C_6$ are in A.P., then n is	(q)	3
(C)	On [0, 2] the maximum value of $f(x) = \max\{x^2, x-1, 3x\}$ is	(r)	0
(D)	If $f(x) = [x] + \sum_{r=1}^{2008} \frac{x+r-[x+r]}{2008}$, then $f(3) =$ ([.] is G.I.F)	(s)	7

54. Match the following:

Column-I		Column-II	
(A)	The shortest distance between the lines $\frac{x-3}{3} = \frac{y-8}{-1} = \frac{z-3}{1}$ and $\frac{x+3}{-3} = \frac{y+7}{2} = \frac{z-6}{4}$ is	(p)	13
(B)	The distance of the point of intersection of the line $\frac{x-2}{3} = \frac{y+1}{4} = \frac{z-2}{12}$ and the plane $x - y + z = 5$ from the point $(-1, -5, -10)$ is	(q)	$\sqrt{\frac{2109}{110}}$
(C)	The length of the perpendicular drawn from the point (5, 4, –1) on the line $\frac{x-1}{2} = \frac{y}{9} = \frac{z}{5}$ is	(r)	2
(D)	The distance between the points P and Q is d and the length of the projection of PQ on the co-ordinate planes are d_1, d_2, d_3 then $d_1^2 + d_2^2 + d_3^2 = kd^2$ when k is	(s)	$3\sqrt{30}$

55. A is a set containing n elements. A subset P of A is chosen at random. The set A is reconstructed by replacing the elements of the subset P. A subset Q of A is again chosen at random the probability that

Column-I		Column-II	
(A)	$P \cap Q = \phi$	(p)	$n(3^{n-1}) / 4^n$
(B)	$P \cap Q$ is a singleton	(q)	$(3/4)^n$
(C)	$P \cap Q$ contain 2 elements.	(r)	${}^{2n}C_n / 4^n$
(D)	$\|P\| = \|Q\|$ where $\|X\|$ = number of elements in X	(s)	$3^{n-2}(n-1)n / 2(4^n)$

SOLUTIONS

1. (a): At the point of minima $f'(x) = 0$ and $f''(x) > 0$

$\Rightarrow$ $f''(x) = -9 + x^2 + f^2(x) > 0 \Rightarrow x^2 + y^2 - 9 > 0$

$\Rightarrow$ point $P(x, f(x))$ lies outside $x^2 + y^2 = 8$

$\Rightarrow$ two tangents are possible.

2. (d): Given equation can be rewritten as

$\sin^2 x + \sin x \cdot \cos x - k \sin x - \sin x - \cos x + k = 0$

$\Rightarrow$ $(\sin x - 1)(\sin x + \cos x - k) = 0$

$\because \quad -\sqrt{2} \le \sin x + \cos x \le \sqrt{2}$ and $\sin x + \cos x = k$

$\Rightarrow -\sqrt{2} \le k \le \sqrt{2} \Rightarrow k = -1, 0, 1, k \ne 1$

3. (d): Let $A \equiv \left(2t_1, \frac{2}{t_1}\right)$, $B \equiv \left(2t_2, \frac{2}{t_2}\right)$, $C \equiv \left(2t_3, \frac{2}{t_3}\right)$

Slope of $AB = \dfrac{\frac{2}{t_1} - \frac{2}{t_2}}{2t_1 - 2t_2} = -\dfrac{1}{t_1 t_2}$

Slope of $AC = -\dfrac{1}{t_1 t_3}$, Slope of $BC = -\dfrac{1}{t_2 t_3}$

Now, $\angle ACB = 90° \Rightarrow \left(-\dfrac{1}{t_1 t_3}\right) \times \left(-\dfrac{1}{t_2 t_3}\right) = -1$

$\Rightarrow \quad t_3^2 = -\dfrac{1}{t_1 t_2}$

$\therefore$ slope of $AB = t_3^2$

Slope of tangent at $C = -\dfrac{1}{t_3^2}$

As (slope of AB) × (slope of tangent at C) = –1

$\Rightarrow$ Angle between them = 90°.

4. (c): $f(x) f(y) - f(xy) = x + y \ \forall \ x, y \in R$ ... (i)

Put $x = 1$, $y = 1$, we get

$(f(1))^2 - f(1) - 2 = 0$

$(f(1) - 2)(f(1) + 1) = 0$

As $f(1) > 0 \Rightarrow f(1) = 2$

Put $y = 1$ in (i), $f(x) \times 2 - f(x) = x + 1$

$\therefore \quad f(x) = x + 1 \Rightarrow f^{-1}(x) = x - 1$

Now, $h(x) = f(x) f^{-1}(x) = x^2 - 1$

$\therefore \quad h(\sin x + \cos x) = (\sin x + \cos x)^2 - 1 = \sin 2x$

As longest interval in which $h(\sin x + \cos x)$ is strictly increasing is π

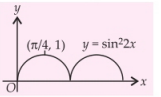
$\Rightarrow$ for $\sin 2x$ it will be $\pi/2$.

5. (b): $\begin{vmatrix} \sin^2 x & \cos^2 x & \alpha \\ \cos^2 x & \alpha & \sin^2 x \\ \alpha & \sin^2 x & \cos^2 x \end{vmatrix} = 0$

$\Rightarrow -(\sin^2 x + \cos^2 x + \alpha)(\sin^4 x + \cos^4 x + \alpha^2 - \alpha \sin^2 x - \alpha \cos^2 x - \sin^2 x \cos^2 x) = 0$

or, $(1 + \alpha)(\alpha^2 - \alpha + 1 - 3\sin^2 x \cos^2 x) = 0$

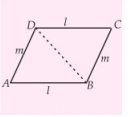
As $\alpha \neq -1 \quad \Rightarrow \quad \alpha^2 - \alpha + 1 = \frac{3}{4}\sin^2 2x$ ... (i)



Now, $\alpha^2 - \alpha + 1 = \left(\alpha - \frac{1}{2}\right)^2 + \frac{3}{4} \geq \frac{3}{4}$ but $\frac{3}{4}\sin^2 2x \leq \frac{3}{4}$

(i) holds only when $\sin^2 2x = 1$
$\Rightarrow \quad \sin 2x = 1$ as $0 < 2x < \pi \quad \Rightarrow \quad x = \pi/4$

6. (d): Let BD be integer and $l \geq m$
$2(l + m) = 40 \quad \Rightarrow \quad l + m = 20$
Possible values of $m = 1, 2, 3, ..., 10$
Note in any triplet of l, m, BD if atleast one is different parallelogram will be non congruent.



Now $l - m < BD < l + m$ (triangle inequality)
$\Rightarrow \quad 20 - 2m < BD < 20$
$\Rightarrow$ Total no. of non congruent parallelograms formed

$$= \sum_{m=1}^{10} (2m-1) = 10^2 = 100$$

[$\because$ Sum of first odd numbers $= n^2$, here $n = 10$]

7. (c): Centre of the circle $|z - 2| = 2$
i.e. 2 lie on $z(1-i) + \bar{z}(1+i) = 4$,
Hence given line $z(1-i) + \bar{z}(1+i) = 4$ pass through the centre of circle *i.e.*, intersect at two points.
$\therefore$ Number of solutions = 2.

8. (a): $\because \quad Q = PAP^T$

$\Rightarrow \quad P^TQ = AP^T \qquad (\because \ P^TP = I)$

$\therefore \quad P^TQ^{2005}P = (P^TQ)Q^{2004}P = AP^T\,Q^{2004}P$
$= A(P^TQ)Q^{2003}\,P = A(AP^T)Q^{2003}\,P$
$= A^2P^T\,Q^{2003}\,P = A^3P^T\,Q^{2002}\,P = ...$
$= A^{2004}\,P^T(QP) = A^{2004}\,P^T(PA)$
$= A^{2005} = \begin{bmatrix} 1 & 2005 \\ 0 & 1 \end{bmatrix}$

9. (b): Required number of ways $= {}^{11}C_5 = 462$

Number of groups $= \frac{30}{6} = 5.$

10. (b): We have, $f(x) = \begin{cases} \frac{a\,|\,x^2 - x - 2\,|}{2 + x - x^2}, & x < 2 \\ b & x = 2 \\ \frac{x - [x]}{x - 2} & x > 2 \end{cases}$

$$= \begin{cases} -\frac{a(x^2 - x - 2)}{2 + x - x^2}, & x < 2 \\ b & x = 2 \\ \frac{x - [x]}{x - 2} & x > 2 \end{cases}$$

$$f(x) = \begin{cases} a, & x < 2 \\ b, & x = 2 \\ \frac{x - [x]}{x - 2} & x > 2 \end{cases}$$

Check at $x = 2$

$$\text{L.H.L} = \lim_{x \to 2^-} f(x) = \lim_{h \to 0} f(2 - h) = \lim_{h \to 0} a = a$$

$$\text{R.H.L} = \lim_{x \to 2+} f(x) = \lim_{h \to 0} f(2 + h)$$

$$= \lim_{h \to 0} \frac{2 + h - [2 + h]}{2 + h - 2} = \lim_{h \to 0} \frac{2 + h - 2}{2 + h - 2} = 1$$

Value of the function $f(2) = b$, so that $f(x)$ is continuous at $x = 2$
$\therefore \quad$ L.H.L = R.H.L = V.F.
$\therefore \quad a = b = 1$

11. (a): Given, $2f(x-1) - f\left(\frac{1-x}{x}\right) = x$ ... (i)

Replacing x by $1/x$, then $2f\left(\frac{1}{x} - 1\right) - f\left(\frac{1 - \frac{1}{x}}{\frac{1}{x}}\right) = \frac{1}{x}$

$$\Rightarrow \quad 2f\left(\frac{1-x}{x}\right) - f(x-1) = \frac{1}{x} \qquad \text{... (ii)}$$

Now, multiplying Eq. (i) by 2 and Eq. (ii) by 1, and then adding, we get

$$3f(x-1) = 2x + \frac{1}{x}$$

Now replacing x by $x + 1$, then

$$f(x) = \frac{1}{3}\left(2(x+1) + \frac{1}{(x+1)}\right)$$

12. (b): $x^2 - 3x + 2 = (x - 1)(x - 2)$
When x^{100} is divided by $x - 1$, the remainder is 1^{100} and when it is divided by $x - 2$, the remainder is 2^{100}.

$$\Rightarrow \quad \frac{x^{100}}{x-1} = q_1(x) + \frac{1}{x-1}, \frac{x^{100}}{x-2} = q_2(x) + \frac{2^{100}}{x-2}$$

On subtracting, we get

$$\frac{-x^{100}}{(x-1)(x-2)} = q_1(x) - q_2(x) + \frac{1}{x-1} - \frac{2^{100}}{x-2}$$

and $\dfrac{x^{100}}{x-2} = (x^{99} + 2x^{98} + 2^2x^{97} + \ldots\ldots + 2^{100}) + \dfrac{2^{100}}{x-2}$

$$\frac{x^{100}}{(x-1)(x-2)} = q_2(x) - q_1(x) + \frac{2^{100}(x-1) - (x-2)}{(x-1)(x-2)}$$

$\Rightarrow$ Remainder is $(2^{100} - 1)x - 2(2^{99} - 1)$

13. (c) : $\tan x - x = 0$

If $x \in \left(0, \dfrac{\pi}{2}\right) \Rightarrow \tan x > x$

$x \in \left(\dfrac{\pi}{2}, \pi\right) \Rightarrow \tan x < x$

$x \in \left(\pi, \dfrac{3\pi}{2}\right) \Rightarrow \tan x \lessgtr x$

$x \in \left(\dfrac{3\pi}{2}, 2\pi\right) \Rightarrow \tan x < x$

14. (c) : $(\vec{u}\ \vec{v}\ \vec{w}) = \left|\vec{u} \cdot (\vec{v} \times \vec{w})\right| = |\vec{u} \cdot (3\hat{i} - 7\hat{j} - \hat{k})|$

$= |\vec{u}|\sqrt{59}\cos\theta$

$\therefore$ Maximum $[\vec{u}\,\vec{v}\,\vec{w}] = \sqrt{59}$ $\quad (\because |\vec{u}| = 1, \cos\theta \le 1)$

15. (a) : Since, $y = f(x)$ and given differential equation $y(1 + xy)dx - xdy = 0$

$\Rightarrow (ydx - xdy) + xy^2dx = 0$

or $\dfrac{ydx - xdy}{y^2} + xdx = 0 \Rightarrow d\left(\dfrac{x}{y}\right) + xdx = 0$

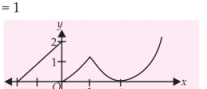
On integrating, then $\dfrac{x}{y} + \dfrac{x^2}{2} = c$ which passes through $(1, 2)$

$\therefore \quad c = 1$

Then, curve is $\dfrac{x}{y} + \dfrac{x^2}{2} = 1$ or $y = \dfrac{2x}{2 - x^2}$

$\therefore \quad f(x) = \dfrac{2x}{2 - x^2}$

16. (c) : $g(f(x))$ is continuous at $x = 0$ and not differentiable for $x = 1$



17. (b): $2^a + 3^b + 5^c = 2^a + (4-1)^b + (4+1)^c$

Case (i) : $a = 2, 3, 4, 5$; $b = 1, 3, 5$; $c = 1, 2, 3, 4, 5$

$4 \times 3 \times 5 = 60$

Case (ii) : $a = 1$; $b = 2, 4$; $c = 1, 2, 3, 4, 5$

$1 \times 2 \times 5 = 10$

No. of ways $= 60 + 10 = 70$

18. (a) : $|\vec{a}_1 + \vec{a}_2 + \ldots\ldots + \vec{a}_n|^2 = 0 \Rightarrow a_1^2 + a_2^2 + \ldots + a_n^2 +$ 2(Sum of the product of a number taken 2 at a time) $= 0$

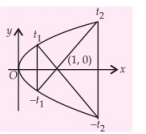
$$\sum_{i=1}^{n} |\vec{a}_i|^2 + 2\sum_{i \le i < j \le n}\sum \vec{a}_i \cdot \vec{a}_j = 0$$

$$\sum_{i \le i < j \le n}\sum \vec{a}_i \cdot \vec{a}_j = \frac{-n}{2}$$

19. (a) : $t_1t_2 = 1$

$(t_1 + t_2)^2 = \dfrac{25}{4}$

$t_1 = \dfrac{1}{2}$, $t_2 = 2$

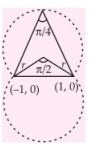


20. (a) : $r = \sqrt{2}$

$l = r\theta$

$2l = 2r\theta$

$= 2\sqrt{2} \times \dfrac{3\pi}{2} = 3\sqrt{2}\pi$



21. (b): Among 1, 2, 3, 4, 5, 6, three numbers are selected in ${}^6C_3$ ways.

No. of onto functions $= 3^n - {}^3C_1\,2^n + {}^3C_2$

$n(E) = {}^6C_3\,(3^n - {}^3C_1\,2^n + {}^3C_2)$

22. (c) : $\displaystyle\int \frac{(\cos 9x + \cos 6x)}{2\cos 5x - 1}dx = A\sin 4x + B\sin x + C$

$$\Rightarrow \int \frac{\left(2\cos\frac{15x}{2}\cos\frac{3x}{2}\right) \times \cos\frac{5x}{2}}{(2\cos 5x - 1)\cos\frac{5x}{2}}dx = A\sin 4x + B\sin x + C$$

$$\Rightarrow \int \frac{2\cos\frac{15x}{2} \cdot \cos\frac{3x}{2}\cos\frac{5x}{2}}{2\cos 5x \cdot \cos\frac{5x}{2} - \cos\frac{5x}{2}}dx = A\sin 4x + B\sin x + C$$

$$\Rightarrow \int \frac{2\cos\frac{15x}{2} \cdot \cos\frac{3x}{2}\cos\frac{5x}{2}}{\cos\frac{15x}{2} + \cos\frac{5x}{2} - \cos\frac{5x}{2}}dx = A\sin 4x + B\sin x + C$$

$$\Rightarrow \int \frac{2\cos\frac{15x}{2} \cdot \cos\frac{5x}{2} \cdot \cos\frac{3x}{2}}{\cos\frac{15}{2}x}dx = A\sin 4x + B\sin x + C$$

$$\Rightarrow \int 2\cos\frac{5x}{2} \cdot \cos\frac{3x}{2}dx = A\sin 4x + B\sin x + C$$

$$\Rightarrow \int (\cos 4x + \cos x)dx = A\sin 4x + B\sin x + C$$

$\Rightarrow \quad \frac{1}{4}\sin 4x + \sin x + C = A\sin 4x + B\sin x + C$

$\Rightarrow \quad A + B = \frac{1}{4} + 1 = \frac{5}{4}$

23. (d): $\lim\limits_{x\to 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

$= \lim\limits_{x\to 0} \frac{e^{\frac{\log(1+x)}{x}} - e}{x} \qquad (\because\ e^{\log M} = M)$

$= \lim\limits_{x\to 0} \frac{e^{\frac{1}{x}\log(1+x)} - e}{x}$

$= \lim\limits_{x\to 0} \frac{e^{\left(x - \frac{x^2}{2} + x^3 - \frac{x^4}{4} + \ldots\ldots\infty\right)}}{x} - e$

$= \lim\limits_{x\to 0} \frac{e^{1 - \frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty}}{x} - e$

$= \lim\limits_{x\to 0} \frac{e\cdot e^{-\frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty}}{x} - e$

$= \lim\limits_{x\to 0} \frac{e\left[e^{-\frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty} - 1\right]}{x}$

$= \lim\limits_{x\to 0} e\left[1 + \frac{\left(-\frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty\right)}{x} + \frac{\left(-\frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty\right)^2}{2!\cdot x} + \ldots\ldots\infty - 1\right]$

$= \lim\limits_{x\to 0} e\left[\frac{x\left(-\frac{1}{2} + \frac{x^2}{3} - \frac{x^3}{4} + \ldots\ldots\infty\right)}{x} + \frac{\left(-\frac{x}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\ldots\infty\right)^2}{2!\cdot x} + \ldots\ldots\infty - 1\right]$

$= e\cdot\left(-\frac{1}{2}\right) + 0 + 0 + \ldots\ldots\infty = -\frac{e}{2}$

24. (c) : No. of irrational terms in expansion of

$\left(\frac{1}{2^4} + \frac{1}{3^{10}}\right)^{55}$

Step 1. Find LCM of 4 and 10 = 20

Step 2. $T_{r+1} = {}^{55}C_r \left(\frac{1}{2^4}\right)^{55-r}\left(\frac{1}{3^{10}}\right)^r = {}^{55}C_r\, 2^{\frac{55-r}{4}}\, 3^{\frac{r}{10}}$

Step 3. $0 \le r \le 55$

Step 4. $\therefore\ r = 0, 20, 40$

When $r = 0$, then $T_1 = {}^{55}C_0\, 2^{\frac{55}{4}}$ which is not rational.

Also 20 and 40 are not the factors of 55.

$\therefore$ No term in the expansion is rational.

and number of terms in the expansion are one more than the exponent of the expansion *i.e.* they are 56 terms.

Hence, three are all 56 terms of the expansion are irrational.

25. (c) : $f(x) + f(1-x) = 1$ ...(i)

and $f(1/2) + f\left(\frac{1}{2}\right) = 1$

$\Rightarrow \quad f\left(\frac{1}{2}\right) = \frac{1}{2}$

which means every term is $\frac{1}{2}$

$\therefore$ Required sum $= \frac{2007}{2} = 1003.5.$

26. (c)

27. (b): In ΔADC, we have $\cos C = \frac{2b}{a}$

$\frac{a^2 + b^2 - c^2}{2ab} = \frac{2b}{a}$

$3b^2 = a^2 - c^2$

28. (b): No. of students who gave wrong answers of atleast i questions $= 2^{n-i}$

No. of students who gave wrong answers of exactly n questions = No. of students who gave wrong answers of atleast n questions $= 2^0 = 1$

No. of students who gave wrong answers to exactly i questions $= 2^{n-i} - 2^{n-(i+1)}$

No. of wrong answers $= \sum_{i=1}^{n-1} i(2^{n-i} - 2^{n-(i+1)}) + (n)$

$= 2047$

$1 + 2 + 2^2 + \ldots. + 2^{n-1} = 2047$

$\Rightarrow \quad 2^n - 1 = 2047 \Rightarrow 2^n = 2^{11} \ \Rightarrow\ n = 11$

29. (a) : $x > 1,\ x^9(x^3 - 1) + x(x^3 - 1) + 1 > 0$

$x < 1,\ (1 - x) + x^4(1 - x^5) + x^{12} > 0$

$x = 1,\ x^{12} - x^9 + x^4 - x + 1 = 1 > 0$

Domain of $f(x)$ is $(-\infty, \infty)$

$g(x) = x^2 + 2x + 4 \ge 3$

Range of $g(x) = [3, \infty)$

30. (c) : Let $w = -1 + 4z$

$|w + 1| = 4|z|; \quad |w + 1| = 2$

Locus of w represents the circle having centre $(-1\ 0)$, radius 2.

31. (a, b, c) : We have $[\vec{p},\vec{q},\vec{r}]=\dfrac{1}{[\vec{a},\vec{b},\vec{c}]}$

$\Rightarrow\ [\vec{p},\vec{q},\vec{r}]>0$

(a) $\because\ x>0, x[\vec{a},\vec{b},\vec{c}]+\dfrac{[\vec{p},\vec{q},\vec{r}]}{x}\geq 2$

(b) Similarly use A.M. $\geq$ G.M.

32. (a, d) : Let $f(x)=4x^3-3x+r$

According to the given condition $f(-1)\,f(0)<0$

$\Rightarrow\ (r-1)\cdot r<0\ \Rightarrow\ r\in(0,1)$.

33. (a, c, d) : (a) L.H.S. $=2^{-m}\int_0^{\pi/2}(\sin 2x)^m\,dx$

$=\dfrac{2^{-m}}{2}\int_0^{\pi}(\sin t)^m\,dt$

$2^{-m}\int_0^{\pi/2}(\sin t)^m\,dt=2^{-m}\int_0^{\pi/2}(\cos t)^m\,dt=$ R.H.S.

(b) $I=\int_0^{\pi}\dfrac{(\pi-x)\sin x}{1+\cos^2 x}\,dx\ \Rightarrow 2I=\pi\int_0^{\pi}\dfrac{\sin x}{1+\cos^2 x}\,dx$

$\Rightarrow\ 2I=2\pi\int_0^{\pi/2}\dfrac{\sin x}{1+\cos^2 x}dx$

$\Rightarrow\ I=\pi\int_0^{\pi/2}\dfrac{\sin x}{1+\cos^2 x}dx=\pi\int_0^{1}\dfrac{dt}{1+t^2}\neq\dfrac{\pi}{2}\int_0^{1}\dfrac{dt}{1+t^2}$

(c) $\int_0^{\pi/2}\dfrac{1-\cos 3x}{1+2\cos x}dx=\int_0^{\pi/2}\dfrac{1+\cos 3x}{2\cos x-1}dx=1$

(d) $I=\int_0^{\pi/2}\dfrac{1+\sin 3x}{1+2\sin x}dx$

Now, using $\int_0^a f(a-x)dx=\int_0^a f(x)dx$

$\int_0^{\pi/2}\dfrac{1+\cos 3x}{2\cos x-1}dx=\int_0^{\pi/2}(2\cos^2 x+\cos x-1)\,dx=1$

$\therefore\ \int_0^{\pi/2}\dfrac{1+\sin 3x}{1+2\sin x}dx=\int_0^{\pi/2}\dfrac{1+\cos 3x}{2\cos x-1}dx$

34. (a, b, c, d) : $AA^T=A^TA=I$.

Also $A^T=A$, so $A^2=I\ \Rightarrow\ A$ is involutory matrix.

$\Rightarrow\ |A^2|=|A|^2=1$ or, $|A|=\pm 1$.

But $|A|=\begin{vmatrix}a&b&c\\b&c&a\\c&a&b\end{vmatrix}=(a+b+c)\begin{vmatrix}1&b&c\\1&c&a\\1&a&b\end{vmatrix}$

$|A|=(a+b+c)\,(ab+bc+ca-a^2-b^2-c^2)$

$|A|=3abc-a^3-b^3-c^3$

but $|A|=-1\ \ \therefore\ a^3+b^3+c^3-3abc=1$

or $=-\dfrac{1}{2}[(a-b)^2+(b-c)^2+(c-a)^2]<0$

$\because\ \ |A|<0$, using $a+b+c=1$

35. (a, b, c) : $g'(x)=f(x)\ \ \because\ \ g(x)=\int_0^x f(t)dt$

$f(x)$ is odd $\ \Rightarrow\ g(x)$ is even

$g(x+2)=\int_0^{x+2}f(t)dt=g(x)+g(2)$

$g(2)=\int_0^2 f(t)dt=\int_0^1 f(t)dt+\int_{-1}^0 f(t)dt=0$ as $f(-t)=-f(t)$

$g(x+2)=g(x)\ \ \therefore$ Period of $g(x)=2$

$g(2n)=n\int_0^2 f(t)dt=0\ \ \ \therefore\ \ g(0)=0$

36. (a, c) : $x<a,\ x^2+2a(x-a)-3a^2=0$

$x^2+2ax-5a^2=0\ \Rightarrow\ x=a(-1+\sqrt{6})$

$x>a,\ x^2-2a(x-a)-3a^2=0$

$x^2-2ax-a^2=0\ \ \Rightarrow\ \ x=a(1-\sqrt{2})$

37. (a, b, c, d) : $b^2+c^2=2\left(l^2+\dfrac{a^2}{4}\right)$

$4l^2=a^2+4bc\cos A$ (use cosine rule)

38. (a, b) : The circle TT' as diameter passes through the focus.

39. (b, d): $(y+b)^2=-2a\left(x-\dfrac{b^2}{2a}+\dfrac{c}{2a}\right)$, it is a parabola. Latus rectum $=2a$.

40. (a, c, d) : Equation at $\left(4\cos\theta,\dfrac{16}{\sqrt{11}}\sin\theta\right)$ to $16x^2+11y^2=256$ is $4x\cos\theta+\sqrt{11}\sin\theta y=16$

$\dfrac{|4\cos\theta-16|}{\sqrt{16\cos^2\theta+11\sin^2\theta}}=4\Rightarrow\cos\theta=\dfrac{1}{2}\ \ \text{or}-\dfrac{5}{2}$

(not possible)

$\therefore\ \ \theta=\pm\dfrac{\pi}{3},\dfrac{5\pi}{3}$

41. (a, b, c, d) : Since planes are parallel planes

$P_1=\dfrac{4}{\sqrt{29}},P_3=\dfrac{8}{\sqrt{29}},P_2=\dfrac{1}{2\sqrt{29}}$

$P_3=16P_2=2P_1,\ P_1+2P_2+3P_3=\sqrt{29}$ or $P_1+8P_2-P_3=0$

42. (a, c) : $2\cos\left(\dfrac{A-C}{2}\right)=\dfrac{\cos\left(\dfrac{A-C}{2}\right)}{\dfrac{1}{2}}=\dfrac{\cos\left(\dfrac{A-C}{2}\right)}{\sin 30^\circ}$.

$$= \frac{2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)}{2\cos\frac{B}{2}\sin\frac{B}{2}}$$

$$\Rightarrow \quad 2\cos\left(\frac{A-C}{2}\right) = \frac{\sin A + \sin C}{\sin B} = \frac{a+c}{b}$$

$$\Rightarrow \quad \frac{a+c}{b} = \frac{a+c}{\sqrt{a^2+c^2-ac}} \Rightarrow \sqrt{a^2+c^2-ac} = b$$

$$\Rightarrow \frac{a^2+c^2-b^2}{2ac} = \frac{1}{2} \Rightarrow \cos B = \frac{1}{2} = \cos 60° \quad \therefore \ \angle B = 60°$$

$\Rightarrow \quad \angle A + \angle C = 120° = 2\angle B \Rightarrow \angle A, \angle B, \angle C$ are in A.P.

43. (b): Point ($r\cos\theta$, $r\sin\theta$), we get

$$r^2 = \frac{3+\sqrt{2}}{3-\sqrt{2}\sin\left(2\theta+\frac{\pi}{4}\right)}, r_{\max} = \sqrt{\frac{3+\sqrt{2}}{3-\sqrt{2}}}, r_{\min} = 1$$

$$\left(\text{as } \sin\left(2\theta+\frac{\pi}{4}\right) \le 1\right)$$

maximum exist when $2\theta + \frac{\pi}{4} = \frac{\pi}{2} \Rightarrow \theta = \frac{\pi}{8}$

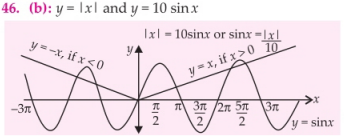
44. (d): Required equation is $\frac{x^2}{3+\sqrt{2}} + y^2 = 1$

$$\Rightarrow \quad x^2(3-\sqrt{2}) + y^2(3+\sqrt{2}) = 3+\sqrt{2}$$

$$\Rightarrow \quad 3(x^2+y^2-1) = \sqrt{2}(x^2-y^2+1)$$

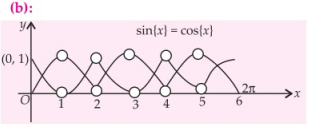
45. (c): Product of length of perpendiculars from foci $= b^2 = 1$

46. (b): $y = |x|$ and $y = 10\sin x$



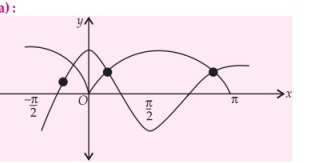
Two curves intersect at 6 points.

47. (b):



Two curves intersect at 6 points.

48. (a):



Two curves intersect at 3 points.

49. (d): $|A|^{(n-1)^2} = |A|^4 = (-1)^4 = 1$

50. (b): $|AB|^{n-1} = |AB|^2 = 24^2$

51. (c): $|\text{adj}(\text{adj}(2A))| = ((-1)^3 |A|^4) = (-1)^{12} |A| = -1$

52. (A)-r, (B)-p, (C)-s, (D)-q

(A) $y = x + \sqrt{1-x}$, Domain : $(-\infty, 1]$

$$\frac{dy}{dx} = 1 - \frac{1}{2\sqrt{1-x}} = 0 \Rightarrow x = \frac{3}{4}$$

$$\frac{d^2y}{dx^2} = \frac{-1}{4(1-x)^{3/2}}$$

$$\left(\frac{d^2y}{dx^2}\right) \text{ at } x = \frac{3}{4} = -2 < 0$$

$\Rightarrow$ y is maximum at $x = 3/4$.

$\therefore \quad y_{\max} = \frac{3}{4} + \frac{1}{2} = \frac{5}{4}$

(B) $y = x^2 e^{-x}$. Domain: R

$$\frac{dy}{dx} = x(2-x)e^{-x} = 0 \Rightarrow x = 0, 2$$

Clearly, $\frac{dy}{dx} < 0$ for $x < 0$

$\frac{dy}{dx} > 0$ for $0 < x < 2$ and $\frac{dy}{dx} < 0$ for $x > 2$

$\therefore \quad$ At $x = 0$, $y_{\min} = 0$

At $x = 2$, $y_{\max} = \frac{4}{e^2}$

(C) $y = x\sqrt{2-x^2}$. Domain : $[-\sqrt{2}, \sqrt{2}]$

$$\frac{dy}{dx} = \frac{2(1-x^2)}{\sqrt{2-x^2}} = 0 \ \Rightarrow \ x = \pm 1$$

For $x < -1$, $\frac{dy}{dx} < 0$ and for $x > -1$, $\frac{dy}{dx} > 0$

$\therefore \quad$ At $x = -1$, $y_{\min} = -1$

For $x < 1$, $\frac{dy}{dx} > 0$ and for $x > 1$, $\frac{dy}{dx} < 0$.

$\therefore \quad$ At $x = 1$, $y_{\max} = 1$

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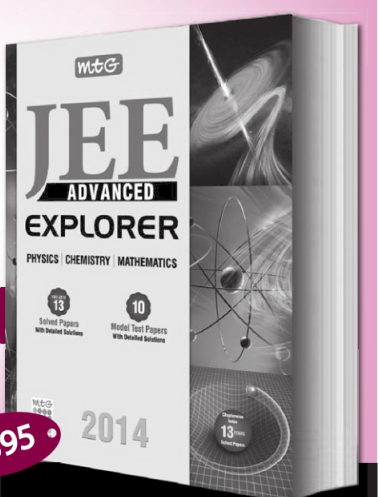

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(D) $y = \dfrac{e^{ax} + e^{-ax}}{2} \Rightarrow \dfrac{dy}{dx} = \dfrac{a(e^{ax} - e^{-ax})}{2} = 0 \Rightarrow x = 0$

$$\frac{d^2y}{dx^2} = \frac{a^2(e^{ax} + e^{-ax})}{2} \text{ and } \left(\frac{d^2y}{dx^2}\right)_{\text{at } x=0} = a^2 > 0.$$

$\therefore$ At $x = 1$, $y_{\min} = 1$.

53. (A)-r, (B)-s, (C)-p, (D)-q

(A) 1, ω, ω^2 are clearly vertices of an equilateral triangle.

(B) $2 \cdot {}^{2n}C_5 = {}^{2n}C_4 + {}^{2n}C_6$

$$\Rightarrow 2 = \frac{{}^{2n}C_4}{{}^{2n}C_5} + \frac{{}^{2n}C_6}{{}^{2n}C_5} \Rightarrow 2 = \frac{5}{2n-4} + \frac{2n-5}{6}$$

$$\Rightarrow 2n^2 - 21n + 49 = 0 \Rightarrow n = 7 \text{ or } n = \frac{7}{2}$$

(C) In the given interval [0, 2], x^2, $x - 1$, $3x$ are all increasing and hence all of them attain their respective maximum values at $x = 2$.

$$\left.\begin{array}{l}\text{At } x = 2, x^2 = 4\\ \text{At } x = 2, x - 1 = 1\\ \text{At } x = 2, 3x = 6\end{array}\right\} \therefore \text{ Ans} = 6$$

(D) $[x + I] = [x] + I$, if I is an integer.

$x + r - [x + r] = x + r - [x] - r$, since r is an integer.

$= x - [x] = \{x\}$

$$\therefore f(x) = [x] + \sum_{r=1}^{2008} \frac{\{x\}}{2008} = [x] + 2008.\frac{\{x\}}{2008}$$

$= [x] + \{x\} = x \quad \therefore f(3) = 3$

54. (A)-s, (B)-p, (C)-q, (D)-r

(A) $\vec{r} = \vec{a} + t \cdot \vec{b}$; $\vec{r} = \vec{c} + s\vec{d}$

Shortest distance between the lines $= \dfrac{|[\vec{c} - \vec{a}\ \vec{b}\ \vec{d}]|}{|\vec{b} \times \vec{d}|}$

(B) (2, –1, 2) satisfies the plane

Find distance between (2, –1, 2) and (–1, –5, –10)

(C) Find d.r's of $A(5, 4, -1)$ and $B(1 + 2k, 9k, 5k)$.

$((2k-4)\hat{i} + (9k-4)\hat{j} + (5k+1)\hat{k}) \cdot (2\hat{i} + 9\hat{j} + 5\hat{k}) = 0$

(D) Projection of PQ on xy-plane

$$= \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$$

Projection of PQ on yz-plane

$$= \sqrt{(y_1 - y_2)^2 + (z_1 - z_2)^2}$$

Projection of PQ on zx-plane

$$= \sqrt{(x_1 - x_2)^2 + (z_1 - z_2)^2}$$

55. (A)-q, (B)-p, (C)-s, (D)-r

If $x_i \in A$ then $x_i \in P$, $x_i \in Q$

$x_i \notin P$, $x_i \notin Q$

$x_i \notin P$, $x_i \in Q$

$x_i \in P$, $x_i \notin Q$

(A) $P \cap Q = \phi \Rightarrow x_i \in P$, $x_i \notin Q$

$x_i \notin P$, $x_i \notin Q$

$x_i \notin P$, $x_i \in Q$

$n(E) = 3^n$, $n(S) = 4^n$

(B) $P \cap Q$ is a singleton

$x_i \in P$, $x_i \in Q$

$n(E) = {}^nC_1(1)3^{n-1}$, $n(S) = 4^n$

(C) $P \cap Q$ contain 2 elements

$n(E) = {}^nC_2\,(1)^2\,3^{n-2}$

(D) $n(P) = n(Q)$

$$\Rightarrow P(E) = \frac{{}^nC_0\,{}^nC_0 + {}^nC_1\,{}^nC_1 + \ldots + {}^nC_n \cdot {}^nC_n}{2^n \cdot 2^n}$$

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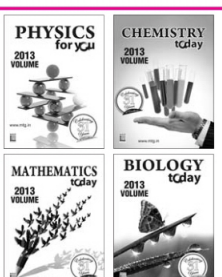

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PAPER-1

SECTION-I

Single Option Correct

This section contains 8 multiple choice questions numbered 1 to 8. Each question has four choices (a), (b), (c) and (d) for its answer, out of which **ONLY ONE** is correct. [3 marks for correct answer and –1 for wrong answer]

1. If $x + ky = 1$ and $x = a$ are the equations of the hypotenuse and a side of a right angled isosceles triangle, then

(a) $k = \pm 1$ (b) $k = \pm a$
(c) $k = \pm 1/a$ (d) $k = \pm 2$

2. Given two points $A \equiv (-2, 0)$ and $B \equiv (0, 4)$. The co-ordinates of a point M lying on the line $x = y$ so that the perimeter of the ΔAMB is least, is

(a) (1, 1) (b) (0, 0)
(c) (2, 2) (d) (3, 3)

3. A ray of light travels along the line $2x - 3y + 5 = 0$ and strikes a plane mirror lying along the line $x + y = 2$. The equation of the straight line containing the reflected ray is

(a) $2x - 3y + 3 = 0$ (b) $3x - 2y + 3 = 0$
(c) $21x - 7y + 1 = 0$ (d) $21x + 7y - 1 = 0$

4. If the inclination of the diameter PP' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to the major axis is θ and PP'^2 is the A.M. of squares of major and minor axes, then $\tan\theta$ is equal to

(a) b/a (b) a/b (c) $\pi/4$ (d) $\pi/6$

5. If a circle of radius 'r' is concentric with ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the common tangent is inclined to major axis at an angle

(a) $\tan^{-1}\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$ (b) $\tan^{-1}\sqrt{\frac{r^2 - a^2}{b^2 - r^2}}$
(c) $\tan^{-1}\sqrt{\frac{r^2 - b^2}{r^2 - a^2}}$ (d) $\tan^{-1}\sqrt{\frac{r^2 - a^2}{r^2 - b^2}}$

6. The angle between the straight lines $\frac{x+1}{2} = \frac{y-2}{5} = \frac{z+3}{4}$ and $\frac{x-1}{1} = \frac{y+2}{2} = \frac{z-3}{-3}$ is

(a) 45° (b) 30°
(c) 60° (d) 90°

7. The distance of the point where the line $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}$ meets the plane $x + 2y + 3z = 14$, from the origin is

(a) $\sqrt{15}$ (b) $\sqrt{14}$
(c) 7 (d) $\sqrt{7}$

8. If the planes $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ passes through one line, then $a^2 + b^2 + c^2 + 2abc =$

(a) ab (b) 1
(c) bc (d) 0

SECTION-II

Assertion-Reason type

This section contains 4 questions numbered 9 to 12. Each question contains Statement-1 (Assertion) and Statement-2 (Reason). Examine the statements carefully and answer the questions according to the instructions given below : [Correct answer : 3 marks, Wrong answer:–1 mark]

(a) If both Statement-1 and Statement-2 are correct and Statement-2 is the proper reason of Statement-1.
(b) If both Statement-1 and Statement-2 are correct and Statement-2 is not the proper reason of Statement-1.
(c) If Statement-1 is correct and Statement-2 is wrong.
(d) If Statement-1 is wrong and Statement-2 is correct.

9. **Statement-1 :** Equation of two angle bisectors of the lines $x = 0$ and $y = 0$ are given by $y = \pm x$.
Statement-2 : Equation of two angle bisectors of the line $a_1x + b_1y + c_1 = 0$, $a_2x + b_2y + c_2 = 0$ is given

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by $\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}}$

10. **Statement-1 :** No transverse common tangent can be drawn to the two circles given by $x^2+y^2=1$ and $(x-1)^2+y^2=1$.
Statement-2 : Two circles are not concentric circles.

11. **Statement-1 :** $l=m=n=1$ represents the direction cosines of z-axis.
Statement-2 : Sum of the squares of direction cosines is equal to 1.

12. **Statement-1 :** For a triangle ABC formed by $A(0, 0)$, $B(1, 0)$, $C(0, 2)$ external angle bisector passing through vertex A is given by $y=-x$.
Statement-2 : Points $B(1, 0)$ and $C(0, 2)$ lie on the same side of the line $y=-x$.

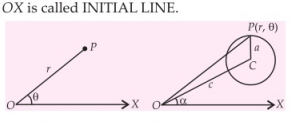
SECTION-III

Linked Comprehension Type

This section contains 2 paragraphs. Based upon each paragraph, 3 multiple choice questions have to be answered. Each question has 4 choices (a), (b), (c) and (d), out of which **ONLY ONE** is correct. [Correct answer : 4 marks, wrong answer :–1 mark]

Paragraph for Q. 13 to Q.15

Suppose OX is a fixed line on which O is a fixed point. Suppose P is a point such that $OP = r$ and anticlockwise angle $XOP = \theta$, then we define (r, θ) as polar co-ordinates of the point P. O is called POLE and line OX is called INITIAL LINE.



Let the polar coordinates of the centre C of a circle be (c, α) and a be the radius of the circle. Let P be any point (r, θ) on the circle. Then in the triangle OPC,

$PC^2 = OC^2 + OP^2 - 2OC \cdot OP \cos OPC$

i.e., $a^2 = c^2 + r^2 - 2c \cdot r \cdot \cos(\theta - \alpha)$

The equation of the circle is therefore,

$r^2 - 2 \cdot cr \cos(\theta - \alpha) + c^2 - a^2 = 0$

We can derive the following corollaries :

1. If the centre of the circle lies on the pole, then $c = 0$ and the equation of the circle becomes $r^2 = a^2$
2. If the circle passes through the pole, then $c = a$ and the equation of the circle becomes $r = 2a\cos(\theta - \alpha)$

13. The straight line $\frac{1}{r} = A\cos\theta + B\sin\theta$ touches the circle $r = 2a\cos\theta$, if

(a) $a^2B^2 + 2aA = 1$ (b) $a^2A^2 + 2aA + a^2B^2 = 1$
(c) $a^2A^2 + 2aB = 1$ (d) $a^2A^2 + 2aA = 1$

14. If a circle passes through the point (r_1, θ_1) and touches the initial line at a distance c from the pole, then its polar equation is $\frac{r^2 - 2cr\cos\theta + c^2}{r\sin\theta} = \lambda$, where $\lambda =$

(a) $r_1\sin\theta_1$ (b) $r_1^2 - 2cr_1\cos\theta + c^2$
(c) $\frac{r_1^2 - 2cr_1\cos\theta_1 + c^2}{r_1\sin\theta_1}$ (d) $\frac{r_1^2 + 2cr_1\cos\theta_1 + c^2}{r_1\sin\theta_1}$

15. O is a fixed point, P is any point on a given circle. OP is joined and on it is taken a point Q such that $OP \cdot OQ$ is a constant quantity k, then the locus of Q is a circle, whose radius is ($OC = c$ and radius of circle $= a$, C being centre)

(a) $\frac{k^2a^2}{|c^2 - a^2|}$ (b) $\frac{\sqrt{1-k^2}c}{|c^2 - a^2|}$
(c) $\left|\frac{ka}{c^2 - a^2}\right|$ (d) $\left|\frac{kc}{c^2 - a^2}\right|$

Paragraph for Q. 16 to Q.18

Suppose that an ellipse and a circle are respectively given by the equations

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (i)

and $x^2 + y^2 + 2gx + 2fy + c = 0$ ....(ii)

The equation

$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda(x^2 + y^2 + 2gx + 2fy + c) = 0$ .....(iii)

represents a curve which passes through the common points of the ellipse (i) and the circle (ii).

We can choose λ so that the equation (iii) represents a pair of straight lines. In general we get three values of λ, indicating three pair of straight lines can be drawn through the points. Also when (iii) represents a pair of straight lines they are parallel to the lines $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \lambda(x^2 + y^2) = 0$ which represents a pair of lines equally inclined to axes (the term containing xy is absent). Hence two straight lines through the points of intersection of an ellipse and any circle make equal angles with the axes. Above description can be applied identically for a hyperbola and a circle.

16. If $\alpha, \beta, \gamma, \delta$ be eccentric angles of the four concyclic points of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, then $\alpha+\beta+\gamma+\delta =$

(a) $(2n+1)\frac{\pi}{2}$ (b) $(2n+1)\pi$

(c) $2n\pi$ (d) $n\pi$

(n is any integer)

17. Let the eccentric angles of three points P, Q and R on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are $\alpha, \frac{\pi}{2}+\alpha$ and $\pi+\alpha$.

A circle through P, Q and R cuts the ellipse again at S, then the eccentric angle of S is

(a) $\pi-3\alpha$ (b) $\frac{3\pi}{2}-3\alpha$

(c) $\frac{\pi}{2}-3\alpha$ (d) $-\frac{\pi}{2}+3\alpha$

18. Suppose two lines are drawn through the common points of intersection of hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ and circle $x^2+y^2+2gx+2fy+c=0$. If these lines are inclined at angles α and β to x-axis, then

(a) $\alpha=\beta$ (b) $\alpha+\beta=\frac{\pi}{2}$

(c) $\alpha+\beta=\pi$ (d) $\alpha+\beta=2\tan^{-1}\left(\frac{b}{a}\right)$

SECTION-IV

Matrix-Match Type

This section contains 2 questions. Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with **ONE OR MORE** statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s ; then the correct darkening of bubbles will look like the above :

	p	q	r	s
A	(p)	(q)	(r)	(s)
B	(p)	(q)	(r)	(s)
C	(p)	(q)	(r)	(s)
D	(p)	(q)	(r)	(s)

[8 marks for correct answer and no negative marking for wrong answer, for correct row 2 marks].

19. Match the following:

Column I		Column II	
(A)	The radius of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having centre at (0, 3) is	(p)	$\frac{2}{\sqrt{3}}$
(B)	If the length of the major axis of an ellipse is three times the length of its minor axis, then its eccentricity is	(q)	$\frac{2\sqrt{2}}{3}$
(C)	The eccentricity of hyperbola whose latus-rectum is 8 and conjugate axis is half the distance between foci, is	(r)	$\sqrt{\frac{17}{20}}$
(D)	The value of m for which $y = mx + 6$ is a tangent to the hyperbola $\frac{x^2}{100}-\frac{y^2}{49}=1$ is	(s)	4

20. Consider the general equation of second degree $ax^2+by^2+2hxy+2gx+2fy+c=0$. If this represents a pair of straight lines, match the two columns in the most accurate sense.

Column I		Column II	
(A)	If (x_1, y_1) is the point of intersection of the two lines then $(ax_1+hy_1)(hx_1+by_1) =$	(p)	$\frac{c}{\sqrt{(a-b)^2+4h^2}}$
(B)	$af^2+bg^2+ch^2 =$	(q)	ab
(C)	The lines are parallel if $h^2 =$	(r)	gf
(D)	Product of perpendiculars from the origin	(s)	$abc+2fgh$

PAPER-2

SECTION-I

Single Option Correct

This section contains 4 multiple choice questions numbered 1 to 4. Each question has 4 choices (a), (b), (c)and (d), out of which **ONLY ONE** is correct. [3 marks for correct answer and –1 for wrong answer]

1. The diagonals of a square are along the pair of lines whose equation is $2x^2-3xy-2y^2=0$. If (2, 1) is a vertex of the square, then another vertex consecutive can be

(a) (1, –2) (b) (1, 4)

(c) (2, –1) (d) (–1, –2)

2. All the chords of curve $3x^2 - y^2 - 2x + 4y = 0$ which subtend a right angle at the origin will pass through
(a) (1, 2) (b) (1, –2)
(c) (2, 1) (d) (0, 0)

3. A circle passing through origin O cuts two straight lines $x - y = 0$ and $x + y = 0$ in points A and B respectively. If abscissae of A and B are the roots of the equation $x^2 + ax + b = 0$, then the equation of the given circle in terms of a and b is
(a) $x^2 + y^2 + ax - by = 0$
(b) $x^2 + y^2 - x\sqrt{4b - a^2} + yb = 0$
(c) $x^2 + y^2 + ax \pm y\sqrt{a^2 - 4b} = 0$
(d) $x^2 + y^2 - ax \pm y\sqrt{a^2 - 4b} = 0$

4. The locus of the mid-points of the chords of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ which are tangent to the ellipse $\frac{x^2}{p^2} + \frac{y^2}{q^2} = 1$ is
(a) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{p^2x^2}{a^4} + \frac{q^2y^2}{b^4}$
(b) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{x^2}{p^2} + \frac{y^2}{q^2}$
(c) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{p^2}{a^2} + \frac{q^2}{b^2}$
(d) $\left(\frac{x^2}{a^2} + \frac{y^2}{b^2}\right)^2 = \frac{p^2}{q^2} + \frac{a^2}{b^2}$

SECTION-II

Multiple Correct type

This section contains 5 multiple choice questions. Each question has 4 choices (a), (b), (c) and (d) for its answer, out of which **ONE OR MORE** is/are correct. [4 marks for correct answer and –1 for wrong answer]

5. The equation $12x^2 - 2y^2 - 6z^2 - 2xy - 8yz + 6zx = 0$ represents a
(a) pair of straight lines
(b) sphere
(c) pair of planes passing through origin
(d) none of these

6. The sphere $x^2 + y^2 + z^2 - 2x - 2y - 2z + 1 = 0$ touches
(a) x-axis (b) y-axis
(c) z-axis (d) none of these

7. If e_1 and e_2 are the eccentricities of the conic sections $16x^2 + 9y^2 = 144$ and $9x^2 - 16y^2 = 144$, then
(a) ${e_1}^2 + {e_2}^2 = 3$ (b) ${e_1}^2 + {e_2}^2 > 3$
(c) ${e_1}^2 + {e_2}^2 < 3$ (d) ${e_1}^2 - {e_2}^2 = 1$

8. The length and foot of perpendicular from the point (7, 14, 5) to the plane $2x + 4y - z = 2$ are
(a) $3\sqrt{21}$ (b) $\sqrt{63}$
(c) (1, 2, 8) (d) (2, 4, –1)

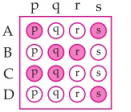
9. A direction line makes angle 60° and 45° with the axes of x and y respectively. The angle it makes with the axis of z is
(a) $\pi/6$ (b) $\pi/3$
(c) $2\pi/3$ (d) $\pi/4$

SECTION-III

Matrix-Match Type

This section contains 3 questions.

Each question contains statements given in two columns, which have to be matched. The statements in Column-I are labelled A, B, C and D, while the statements in Column-II are labelled p, q, r and s. Any given statement in Column-I can have correct matching with ONE OR MORE statement(s) in Column-II. The appropriate bubbles corresponding to the answers to these questions have to be darkened as illustrated in the following example. If the correct matches are A-p and s ; B-q and r; C-p and q; and D-s ; then the correct darkening of bubbles will look like the above. [8 marks for correct answer and no negative marking for wrong answer and each row 2 marks]



10. Match the following:

Column I		Column II	
(A)	The number of common tangents to the circles $x^2 + y^2 - x = 0$ and $x^2 + y^2 + x = 0$ is	(p)	$6/\sqrt{5}$
(B)	The length of the chord cut off by $y = 2x + 1$ from the circle $x^2 + y^2 = 2$ is	(q)	3
(C)	If a circle passes through the points of intersection of the co-ordinate axes with the lines $\lambda x - y + 1 = 0$ and $x - 2y + 3 = 0$, then the value of λ is	(r)	2

(D)	For the circle $x^2 + y^2 = 36$, the length of the chord lying along the line $3x + 4y - 15 = 0$ is	(s)	$6\sqrt{3}$

11. Match the following:

Column I		Column II	
(A)	The normal chord at a point 't' on the parabola $y^2 = 4x$ subtends a right angle at the vertex. Then t^2 is equal to	(p)	4
(B)	The area of the triangle inscribed in the curve $y^2 = 4x$, the parameter of coordinates of whose vertices are 1, 2 and 4 is (In parametric form $(at^2, 2at)$, t is the parameter)	(q)	2
(C)	The number of distinct normals possible from $\left(\frac{11}{4}, \frac{1}{4}\right)$ to the parabola $y^2 = 4x$ is	(r)	3
(D)	If the normal at $(a, 2a)$ on $y^2 = 4ax$ meets the curve again at $(at^2, 2at)$, then the value of $\lvert t - 1 \rvert$ is	(s)	6

SECTION-IV

Integer Answer Type

This section contains 8 questions. The answer to each of the questions is a single digit integer, ranging from 0 to 9. The appropriate bubbles below the respective question numbers in the ORS have to be darkened. For example, if the correct answers to questions numbers X, Y, Z, W are 6, 0, 9, 2 respectively then the correct darkening of bubbles will look like the following: [4 marks for correct answer and –1 for wrong answer]

12. The locus of the centre of the circle for which one end of diameter is (3, 3) while the other end lies on the line $x + y = 4$ is $x + y = k$, then k equals

13. The greatest distance of the point (10, 7) from the circle $x^2 + y^2 - 4x - 2y - 20 = 0$ is 5α, then α equals

14. Angle between the tangents drawn from (1, 4) to the parabola $y^2 = 4x$ is π/m, where m equals

15. If the straight line $y = 2x + c$ is a tangent to the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$ then $\lvert c \rvert$ equals

16. The foci of the ellipse $\frac{x^2}{16} + \frac{y^2}{b^2} = 1$ and the hyperbola $\frac{x^2}{144} - \frac{y^2}{81} = \frac{1}{25}$ coincide, then value of b^2 equals

17. The intersection of the sphere $x^2 + y^2 + z^2 + 7x - 2y - z = 13$ and $x^2 + y^2 + z^2 - 3x + 3y + 4z - 8 = 0$ is the same as the intersection of one of the sphere and the plane $ax + by + cz = d$, Then the least value of $a + b + c + d$ is

18. The distance of point (3, 8, 2) from the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{3}$ measured parallel to the plane $3x + 2y - 2z + 15 = 0$ is

19. The shortest distance between z-axis and the lines $x + y + 2z - 3 = 0$, $2x + 3y + 4z - 4 = 0$ is

ANSWERS KEYS

Paper-1

1. (a) **2.** (b) **3.** (b) **4.** (a) **5.** (a) **6.** (d)
7. (b) **8.** (b) **9.** (a) **10.** (b) **11.** (d) **12.** (a)
13. (a) **14.** (c) **15.** (c) **16.** (c) **17.** (c) **18.** (c)
19. (A) → (s); (B) → (q); (C) → (p), (D) → (r)

20. (A) → (r); (B) → (s); (C) → (q), (D) → (p)

Paper-2

1. (a) **2.** (b) **3.** (c) **4.** (a) **5.** (c)
6. (a, b, c, d) **7.** (c) **8.** (a, c) **9.** (b, c)

10. (A) → (q); (B) → (p); (C) → (r), (D) → (s)

11. (A) → (q); (B) → (s); (C) → (r), (D) → (p)

12. (5) **13.** (3) **14.** (3) **15.** (6) **16.** (7) **17.** (1)
18. (7) **19.** (2)

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PRACTICE PAPER 2014

JEE Advanced

SECTION-I

Single Correct Answer Type

For each correct answer 3 marks will be awarded and each incorrect choice 1 mark will be deducted. Each question has single correct answer only.

1. The exhaustive set of values of α^2 such that there exists a tangent to the ellipse $x^2 + \alpha^2 y^2 = \alpha^2$ such that the portion of the tangent intercepted by the hyperbola $\alpha^2 x^2 - y^2 = 1$ subtends a right angle at the centre of the curves, is

(a) $\left[\frac{\sqrt{5}+1}{2}, 2\right]$ (b) $(1, 2]$

(c) $\left[\frac{\sqrt{5}-1}{2}, 1\right)$

(d) $\left[\frac{\sqrt{5}-1}{2}, 1\right) \cup \left(1, \frac{\sqrt{5}+1}{2}\right]$

2. The value of ${}^{404}C_4 - {}^4C_1 {}^{303}C_4 + {}^4C_2 {}^{202}C_4 - {}^4C_3 {}^{101}C_4$ is equal to

(a) $(401)^4$ (b) $(101)^4$

(c) 0 (d) $(201)^4$

3. The range of the function $f(x) = \frac{\sin x}{x} + \frac{x}{\tan x}$ in $\left(0, \frac{\pi}{2}\right)$ is

(a) $(0, \infty)$ (b) $\left(\frac{1}{\pi}, 2\right)$

(c) $(2, \infty)$ (d) $\left(\frac{2}{\pi}, 2\right)$

4. The number of solutions of $e^{2x} = 9x^2$ is

(a) zero (b) two

(c) three (d) four

5. If α, β, γ are the roots of the equation $x^3 + ax^2 + bx + c = 0$ where $(a \neq 0)$ and if the system of equations $\alpha u + \beta v + \gamma w = 0$, $\beta u + \gamma v + \alpha w = 0$, $\gamma u + \alpha v + \beta w = 0$ has non-trivial solutions, then $a^2 =$

(a) b (b) $2b$

(c) $3b$ (d) none of these

6. The position vector of point of intersection of the planes $\vec{r} \cdot (\hat{n}_2 \times \hat{n}_3) = 1$, $\vec{r} \cdot (\hat{n}_3 \times \hat{n}_1) = 2$, and $\vec{r} \cdot (\hat{n}_1 \times \hat{n}_2) = 3$, $([\hat{n}_1, \hat{n}_2, \hat{n}_3] \neq 0)$ is

(a) $\frac{\hat{n}_1 + 2\hat{n}_2 + 3\hat{n}_3}{[\hat{n}_1, \hat{n}_2, \hat{n}_3]}$ (b) $\frac{\hat{n}_1 + 2\hat{n}_2 + 3\hat{n}_3}{[\hat{n}_1, \hat{n}_2, \hat{n}_3]^2}$

(c) $\frac{\hat{n}_2 \times \hat{n}_3 + 2\hat{n}_3 \times \hat{n}_1 + 3\hat{n}_1 \times \hat{n}_2}{[\hat{n}_1, \hat{n}_2, \hat{n}_3]}$

(d) $\frac{\hat{n}_2 \times \hat{n}_3 + 2\hat{n}_3 \times \hat{n}_1 + 3\hat{n}_1 \times \hat{n}_2}{[\hat{n}_1, \hat{n}_2, \hat{n}_3]^2}$

7. $\int_0^{n\pi} \ln|a\sin x + b\cos x|\, dx \; (n \in N)$ is equal to

(a) $\frac{n\pi}{2}\ln(a^2 + b^2)$ (b) 0

(c) $\frac{n\pi}{2}\ln\left(\frac{a^2+b^2}{2}\right)$ (d) $\frac{n\pi}{2}\ln\left(\frac{a^2+b^2}{4}\right)$

8. If the parabola $y^2 = \sqrt{2}ax \; (a > 0)$ meets the rectangular hyperbola $xy = 4\sqrt{2}$ orthogonally then the value of 'a' is

(a) $\sqrt{2}$ (b) 2 (c) $2\sqrt{2}$ (d) 4

9. The range of values of $\cos^{-1}x \cdot \cos^{-1}(-x)$ is

(a) $[0, \pi^2]$ (b) $\left[0, \frac{\pi^2}{4}\right]$

(c) $\left(0, \frac{\pi^2}{2}\right)$ (d) $\left[\frac{-\pi^2}{2}, \frac{\pi^2}{2}\right]$

10. Let 'f' and 'g' be two real valued functions such that $f(2-x) = 2 - f(x)$ and $f(x) - g(x) = 1$, then

(a) $f(x)$ is symmetrical about the line $x = 1$.

(b) $g(x)$ is symmetrical about the line $x = 1$.

(c) $f(x)$ is symmetrical about the point $(1, 0)$.

(d) $g(x)$ is symmetrical about the point $(1, 0)$.

11. Let H, I and O be respectively the orthocentre, incentre and circumcentre of the triangle ABC. If $\angle HAI = \alpha$ and $\angle IAO = \beta$, then

(a) $2\alpha = \beta$ (b) $\alpha = 2\beta$

(c) $\alpha = 4\beta$ (d) $\alpha = \beta$

12. If $\vec{a}, \vec{b}$ are any two vectors of magnitudes 2, 3 respectively, such that $|2(\vec{a} \times \vec{b})| + |3(\vec{a} \cdot \vec{b})| = k$, then the maximum value of k is

(a) $\sqrt{13}$ (b) $2\sqrt{13}$

(c) $6\sqrt{13}$ (d) $10\sqrt{13}$

SECTION-II

Multiple Correct Answer Type

For each correct answer 5 marks will be awarded and each incorrect choice 2 marks will be deducted.

13. The number $\frac{5^k + 3}{4}(k \in N)$, when divided by 10, may leave remainder

(a) 2 (b) 6 (c) 7 (d) 8

14. For the equation $|x^2 - a| = \cos^{-1}(\cos x)$ to have some solution, the value of 'a' can be

(a) 1/2 (b) 1/4 (c) $-1/2$ (d) $-1/4$

15. The function $f(x) = a(x^2 - 1)(ax + b)(a \neq 0)$ has

(a) a local maxima at certain $x \in R^+$

(b) a local minima at certain $x \in R^+$

(c) a local maxima at certain $x \in R^-$

(d) a local minima at certain $x \in R^-$

16. Which of the following is/are correct?

(a) Between any two roots of $e^x \cos x = 1$ there exists atleast one root of $\tan x = 1$

(b) Between any two roots of $e^x \sin x = 1$ there exists atleast one root of $\tan x = -1$

(c) Between any two roots of $e^x \cos x = 1$ there exists atleast one root of $e^x \sin x = 1$

(d) Between any two roots of $e^x \sin x = 1$ there exists atleast one root of $e^x \cos x = 1$

17. Let $f(x) = \frac{e^x}{1 + x^2}$ and $g(x) = f'(x)$, then

(a) $g(x)$ has four points of local extremum

(b) $g(x)$ has two points of local extremum

(c) $g(x)$ has a point of local minimum at $x = 1$

(d) $g(x)$ has a point of local maximum at some $x \in (-1, 0)$

18. The value of $\sum_{k=0}^{7} \left[\frac{\binom{7}{k}}{\binom{14}{k}} \sum_{r=k}^{14} \binom{r}{k}\binom{14}{r} \right]$, where $\binom{n}{r}$ denotes ${}^nC_r$, is

(a) 6^7 (b) greater than 7^6

(c) 8^7 (d) greater than 7^8

19. If $P(\alpha, \beta)$, the point of intersection of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{a^2(1 - e^2)} = 1$ and the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{a^2(E^2 - 1)} = \frac{1}{4}$, is equidistant from the foci of the two curves (all lying in the right of y-axis), then

(a) $2\alpha = a(2e + E)$ (b) $a - e\alpha = E\alpha - a/2$

(c) $E = \frac{\sqrt{e^2 + 24} - 3e}{2}$ (d) $E = \frac{\sqrt{e^2 + 12} - 3e}{2}$

20. The solution set of $|\sin x| \leq |\cos 2x|$ contains

(a) $\bigcup_{n \in I} \left\{ \left[n\pi - \frac{\pi}{6}, n\pi + \frac{\pi}{6} \right] \right\}$

(b) $\bigcup_{n \in I} \left\{ n\pi + \frac{\pi}{2} \right\}$

(c) $\bigcup_{n \in I} \left\{ \left[n\pi - \frac{\pi}{8}, n\pi + \frac{\pi}{8} \right] \right\}$

(d) $\bigcup_{n \in I} \left\{ \left[n\pi - \frac{\pi}{4}, n\pi + \frac{\pi}{4} \right] \right\}$

SECTION-III

Matrix Match Type

For each correct answer 6 marks will be awarded. There is no negative marking for these questions. There could be a single answer or multiple answers.

21. Let $f : R \to R$, $f(x) = 2x^3 - 3(a + 2)x^2 + 12ax - 7$, $-4 \leq a \leq 6$, $a \in I$. For $f(x)$

Column I		Column II	
(A)	to have only one real root	(p)	$a = -1$
(B)	to have two equal roots	(q)	$a \in \{0, 1, 2, 3, 4, 5\}$
(C)	to be invertible	(r)	$a \in \{-4, -3, -2, 6\}$
(D)	to have three real and distinct roots	(s)	$a = 2$

22. Match the following:

	Column I		Column II
(A)	$f : [0, \pi/2] \to [0, 1]$; $f(x) = \lvert \sin x \rvert$	(p)	one-one
(B)	$f : (-\infty, -1] \to (0, e^5]$; $f(x) = e^{x^3 - 3x + 2}$	(q)	many-one
(C)	$f : R \to R$; $f(x) = x^2 + x$	(r)	onto
(D)	$f : R \to R$; $f(x) = 2x + \sin x$	(s)	into

23. Match the following:

	Column I		Column II
(A)	If $\frac{a_1}{a_2} = \frac{b_1}{b_2}$, then $(a_1x + b_1y + c_1)$ $(a_2x + b_2y + c_2) + k = 0$ $(k \neq 0)$ represents	(p)	a parabola
(B)	If $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$, then $(a_1x + b_1y + c_1)$ $(a_2x + b_2y + c_2) + k = 0$ $(k \neq 0)$ represents	(q)	a pair of lines
(C)	Locus of a point moving such that its distances from the point (–13, 7) and the line $17x + 29y - 18 = 0$ are always equal	(r)	a straight line
(D)	Locus of a point moving such that the ratio of its distances from the point (3, 11) and the line $14x - 5y + 13 = 0$ is always 2	(s)	a hyperbola

24. Match the following:

	Column I		Column II
(A)	If ω be an imaginary cube root of unity and $x_1, x_2, x_3 \in R$, $\sum_{r=1}^{3} \frac{1}{x_r + \omega} = 2\omega^2$, $\sum_{r=1}^{3} \frac{1}{x_r + \omega^2} = 2\omega$ then $\sum_{r=1}^{3} \frac{1}{x_r + 1}$ is equal to	(p)	20
(B)	If z_1, z_2, z_3 be the vertices A, B, C respectively of a triangle ABC such that $\lvert z_1 \rvert = \lvert z_2 \rvert = \lvert z_3 \rvert$ and $\lvert z_1 + z_2 \rvert = \lvert z_3 - z_2 \rvert$, then tan C is	(q)	2
(C)	The term independent of x in the expansion of $(18 + x + 2x^2)\left(\frac{3}{2}x^2 - \frac{1}{3x}\right)^9$	(r)	1
(D)	If C_r stands for ${}^nC_r$, and $\sum_{r=1}^{n} \frac{r.C_r}{C_{r-1}} = 210$, then n is equal to	(s)	7

SOLUTIONS

1. (a) : Homogenising, we get

$$\alpha^2x^2 - y^2 = \left(\frac{x}{\alpha}\cos\theta + y\sin\theta\right)^2.$$

Coefficient of x^2 + coefficient of $y^2 = 0$

$$\Rightarrow \alpha^2 = \frac{\cos^2\theta}{\alpha^2} - 1 - \sin^2\theta = 0$$

$$\Rightarrow 0 \le \cos^2\theta = \frac{\alpha^2(2-\alpha^2)}{\alpha^2 - 1} \le 1 \Rightarrow \alpha^2 \in \left[\frac{\sqrt{5}+1}{2}, 2\right]$$

2. (b): The given expression is the coefficient of x^4 in

$${}^4C_0(1+x)^{404} - {}^4C_1(1+x)^{303} + {}^4C_2(1+x)^{202} - {}^4C_3(1+x)^{101} + {}^4C_4$$

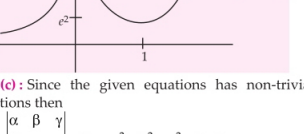
i.e., in $((1+x)^{101} - 1)^4$

i.e., in $({}^{101}C_1 x + {}^{101}C_2x^2 + \ldots)^4 = (101)^4$

3. (d) : $\frac{\sin x}{x}$ and $\frac{x}{\tan x}$ both are decreasing functions of x in $\left(0, \frac{\pi}{2}\right)$ so, maximum value, when $x \to 0$ is 1 + 1 = 2 minimum when $x \to \frac{\pi}{2}$ is $\frac{2}{\pi}$.

4. (c) : Let $f(x) = \frac{e^{2x}}{x^2}$, $f'(x) = \frac{2e^{2x}(x-1)}{x^3}$

$f(x) = 9$ has three solutions.



5. (c) : Since the given equations has non-trivial solutions then

$$\begin{vmatrix} \alpha & \beta & \gamma \\ \beta & \gamma & \alpha \\ \gamma & \alpha & \beta \end{vmatrix} = 0 \Rightarrow \alpha^3 + \beta^3 + \gamma^3 = 3\alpha\beta\gamma$$

We have $a^3 - 3ab = 0$

6. (a) :

$$\vec{r} = \frac{(\vec{r}\,.\,(\hat{n}_2 \times \hat{n}_3))\hat{n}_1 + (\vec{r}\,.\,(\hat{n}_3 \times \hat{n}_1))\hat{n}_2 + (\vec{r}\,.\,(\hat{n}_1 \times \hat{n}_2))\hat{n}_3}{[\hat{n}_1, \hat{n}_2, \hat{n}_3]}$$

$$= \frac{\hat{n}_1 + 2\hat{n}_2 + 3\hat{n}_3}{[\hat{n}_1, \hat{n}_2, \hat{n}_3]}$$

7. (d): $\int_0^{n\pi} \ln|a\sin x + b\cos x|\,dx = n\int_0^{\pi} \ln|r\sin(x+\alpha)|\,dx$

$= n\int_0^{\pi} \ln|r\sin x|\,dx = 2n\left[\int_0^{\pi/2} \ln r\,dx + \int_0^{\pi/2} \ln\sin x\,dx\right]$

$= 2n\left[\frac{\pi}{2}\ln r - \frac{\pi}{2}\ln 2\right] = \frac{n\pi}{2}\left[\ln\left(\frac{a^2+b^2}{4}\right)\right]$

8. (c): $y^2 = \sqrt{2}ax$ and $xy = 4\sqrt{2}$. Solving, we get

$x = 2\sqrt{2}a^{-\frac{1}{3}}, y = 2a^{\frac{1}{3}}$,

$y_1' = \frac{a}{\sqrt{2}y} = \frac{a^{2/3}}{2\sqrt{2}}$, $y_2' = -\frac{4\sqrt{2}}{x^2} = -\frac{a^{2/3}}{\sqrt{2}}$, $y_1' \cdot y_2' = -1$

$\Rightarrow \frac{a^{4/3}}{4} = 1 \Rightarrow a = 2^{3/2} = 2\sqrt{2}$

9. (b):

$0 \le \sqrt{\cos^{-1}x.\cos^{-1}(-x)} \le \frac{\cos^{-1}x + \cos^{-1}(-x)}{2} = \frac{\pi}{2}$

10. (d): $f(2-x) - 1 = 1 - f(x) \Rightarrow g(2-x) = -g(x)$

replacing x by $x+1$ we get, $g(1-x) = -g(1+x)$.

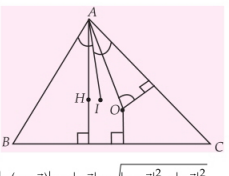
So, g is symmetrical about the point (1, 0).

11. (d): $\angle BAH = 90° - B = \angle OAC$

$\angle BAI = A/2 = \angle IAC$

So, $\angle HAI = A/2 - (90° - B) = \angle IAO$

$\Rightarrow \alpha = \beta$



12. (c): $\left|2(\vec{a}\times\vec{b})\right| + 3\left|\vec{a}.\vec{b}\right| \le \sqrt{\left|\vec{a}\times\vec{b}\right|^2 + \left|\vec{a}.\vec{b}\right|^2}$

$\Rightarrow \sqrt{4+9} \le 6\sqrt{13}$

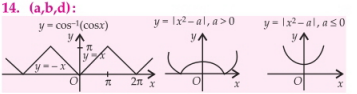
13. (a,c): $\frac{5^k+3}{4} = \frac{5^k-5+8}{4} = \frac{5(5^{k-1}-1)}{5-1} + 2$

$= 5 + 5^2 + 5^3 + ... + (k-1)$ terms $+ 2$

So if $k-1$ = even, the last digit is 2.

If $k-1$ = odd, the last digit is $5 + 2 = 7$.

14. (a,b,d):



For $a > 0$, there is always a solution. For $a \le 0$, the solution exists for all $a \in [b, 0]$ where b is the number when $y = |x^2 - b|$ touches $y = \cos^{-1}(\cos x)$.

So, $y_1 = y_2$ and $y_1' = y_2' \Rightarrow x = x^2 - b$ and $1 = 2x \Rightarrow b = -\frac{1}{4}$.

So finally $a \in \left[-\frac{1}{4}, \infty\right)$.

15. (b, c): $f(x) = a^2x^3 + abx^2 - a^2x - ab$.

Coefficient of x^3 is +ve.

$f'(x) = 3a^2x^2 + 2abx - a^2$. Product of roots is –ve

So, $f(x)$ has two points of maxima/minima and maxima at R^- and minima at R^+.

16. (a, b, c): (a) Let $f(x) = e^x \cos x - 1$

$f'(x) = e^x(\cos x - \sin x) = 0$

$\Rightarrow \tan x = 1$. Between two roots of $f(x)$ (Rolle's theorem)

(b) $g(x) = e^x \sin x - 1$, $g'(x) = e^x(\sin x + \cos x) = 0$

$\Rightarrow \tan x = -1$ between two roots of $g(x)$.

(c) $h(x) = e^{-x} - \cos x$, $h'(x) = -e^{-x} + \sin x = 0$

$\Rightarrow e^{-x} = \sin x$ between two roots of $h(x)$.

17. (b, c, d): $f(x) = \frac{e^x}{1+x^2}$, $g(x) = f'(x) = \frac{(x-1)^2e^x}{(1+x^2)^2}$,

$g'(x) = \frac{(x-1)(x^3 - 3x^2 + 5x + 1)}{(x^2+1)^3}e^x$

Let $h(x) = x^3 - 3x^2 + 5x + 1$, $h'(x) = 3x^2 - 6x + 5$, $D < 0$ so $h(x)$ has only one real root. Also $g(-1)\ g(0) < 0$. So the root $\in (-1, 0)$. Clearly $g(x)$ has two points of extremum. Maxima at $x \in (-1, 0)$ and minima at $x = 1$.

18. (a, b): $\sum_{k=0}^{7}\left(\frac{{}^7C_k}{{}^{14}C_k}\sum_{r=k}^{14}{}^rC_k \cdot {}^{14}C_r\right)$

$= \sum_{k=0}^{7}\left(\frac{{}^7C_k}{|14}\times\frac{|k\,|14-k}{}\sum_{r=k}^{14}\frac{|r}{|k\,|r-k}\cdot\frac{|14}{|r\,|14-r}\right)$

$= \sum_{k=0}^{7}\left({}^7C_k\sum_{r=k}^{14}{}^{14-k}C_{r-k}\right) = \sum_{k=0}^{7}{}^7C_k \cdot 2^{14-k} = 2^{14}\sum_{k=0}^{7}{}^7C_k\left(\frac{1}{2}\right)^k$

$= 2^{14}\cdot\left(1+\frac{1}{2}\right)^7 = 6^7 > 7^6$

19. (b,c): $S_1P = S_2P \Rightarrow a - e\alpha = E\alpha - \left(\frac{a}{2}\right)$.

Also, $\alpha = \frac{ae + \frac{aE}{2}}{2}$

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Eliminating α, we get

$$E^2 + 3eE + (2e^2 - 6) = 0 \Rightarrow E = \frac{\sqrt{e^2+24} - 3e}{2}$$

20. (a,b,c) : $|\cos 2x|^2 \geq |\sin x|^2$

$\Rightarrow \cos^2 2x \geq \dfrac{1-\cos 2x}{2}$

$\Rightarrow (\cos 2x + 1)(2\cos 2x - 1) \geq 0$

$\Rightarrow$ either $\cos 2x = -1$ or $\cos 2x \geq \dfrac{1}{2} \Rightarrow$ either

$2x \in 2n\pi + \pi$ *or* $\left[2n\pi - \dfrac{\pi}{3}, 2n\pi + \dfrac{\pi}{3}\right], n \in I$

21. (A)-(q), (B)-(p), (C)-(s), (D)-(r)

$f(x) = 2x^3 - 3(a+2)x^2 + 12ax - 7$

$f'(x) = 6[x^2 - (a+2)x + 2a] = 6(x-a)(x-2)$

For $f(x)$ to be invertible, $f'(x) \geq 0 \ \forall \ x \in R \Rightarrow a = 2$

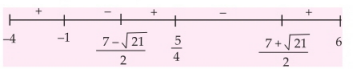
For $f(x)$ to have three real and distinct roots,

$a \neq 2$ and $f(a)\, f(2) < 0$

$\Rightarrow (2a^3 - 3(a+2)a^2 + 12a^2 - 7)(16 - 12(a+2) + 24a - 7) < 0$

$\Rightarrow (a^3 - 6a^2 + 7)(4a - 5) > 0$

$\Rightarrow (a+1)(a^2 - 7a + 7)(4a - 5) > 0$



$\Rightarrow a = -4, -3, -2, 6$

For $f(x)$ to have two equal roots, $a \neq 2$ and $f(a)\, f(2) = 0$

$\Rightarrow a = -1$

For $f(x)$ to have only one real root $a = 2$ or $f(a)\, f(2) > 0$

$\Rightarrow a = 0, 1, 2, 3, 4, 5.$

22. (A)-(p, r); (B)-(p, s); (C)-(q, s); (D)-(p, r)

23. (A)-(q), (B)-(s), (C)-(r), (D)-(q)

(A) If $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2}$, $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) = 0$ represents a pair of parallel lines so $\Delta = 0$ and $h^2 = ab$.
Now in $(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + k = 0$ $(k \neq 0)$, $\Delta' = 0 + abk - kh^2 = k(ab - h^2) = 0$, so it is again a pair of lines.

(B) If $\dfrac{a_1}{a_2} \neq \dfrac{b_1}{b_2}$, $h^2 > ab$ $\Delta' \neq 0$, so it is a hyperbola.

(C) The point is on the line. So, the locus is a straight line perpendicular to the given line through the given point.

(D) The point is on the line. So the locus is a pair of lines making angle $\sin^{-1}(1/2)$ *i.e.* 30° with the given lines and passing through the given point.

24. (A)-(q), (B)-(r), (C)-(s), (D)-(p)

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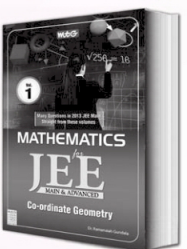

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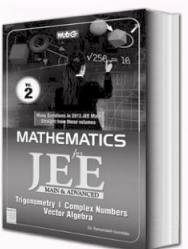

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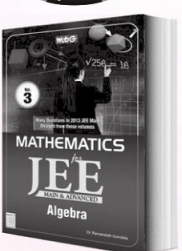

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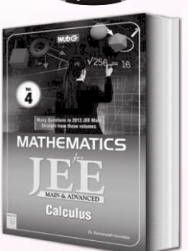

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MATHS MUSING

SOLUTION SET-136

1. (c) : $N=\left(\sqrt{3}+1\right)^{2014}+\left(\sqrt{3}-1\right)^{2014}$

$=2^{1007}\left[\left(2+\sqrt{3}\right)^{1007}+\left(2-\sqrt{3}\right)^{1007}\right]$

$=2^{1009}\left[2^{1006}+\binom{1007}{2}2^{1004}.3+...+\binom{1007}{1006}3^{503}\right]$

$\therefore \quad 2^{1009}=2\cdot 16^{252}$ divides N. $\Rightarrow \quad p=252$

2. (b) : $9N=(2\cdot 10^{2013}+1)^2=4\cdot 10^{4026}+4\cdot 10^{2013}+1$

$\therefore \quad N=\dfrac{4(10^{4026}-1)}{(10-1)}+\dfrac{4(10^{2013}-1)}{(10-1)}+1$

= 44.... 488...89 (2013 times 4, 2013 times 8 and one 9), sum of the digits of N = 24157

3. (c) : $\left(\dfrac{a+b+c+d}{4}\right)^2\le\dfrac{a^2+b^2+c^2+d^2}{4}$

$(8-p)^2\le 4(16-p^2) \Rightarrow p\le\dfrac{16}{5}$

4. (b) : Replacing $f(x)$ by x and x by $g(x)$.

$x=\int_0^{g(x)}\dfrac{dt}{\sqrt{1+t^3}}$. Differentiating, $1=\dfrac{g'}{\sqrt{1+g^3}}$

or $(g')^3=1+g^3$. Differentiating, $g''=\dfrac{3}{2}g^2$

5. (b) : Required Probability

$=\binom{7}{3}4!\left(\dfrac{1}{2!}-\dfrac{1}{3!}+\dfrac{1}{4!}\right)\div 7!$

$=\dfrac{1}{3!}\left(\dfrac{1}{2}-\dfrac{1}{6}+\dfrac{1}{24}\right)=\dfrac{1}{16}$

6. (b, c) : $AX=\lambda X=(A-\lambda I)X=0$

$X\ne 0\Rightarrow\begin{vmatrix}4-\lambda & 6 & 6\\ 1 & 3-\lambda & 2\\ -1 & -5 & -2-\lambda\end{vmatrix}=0$

$\Rightarrow (\lambda-1)(\lambda-2)^2=0$

$\therefore \quad \lambda=1, 2.$

7. (c) : $2a$ and $2b$ are major and minor axes. Centre C (4, 4), foci S(6, 2) and S_1 (2, 6)

$b^2=2\times 6=12.$

$SS_1=2\sqrt{a^2-b^2}$, $2(6-2)^2=4(a^2-12)$

$a^2=20$, $e=\sqrt{\dfrac{2}{5}}$. Directrix is $x-y-10=0$

Equation of ellipse $(x-6)^2+(y-2)^2=\dfrac{2}{5}\dfrac{(x-y-10)^2}{2}$

$y=0\Rightarrow x=5$, $OA=5$

8. (d) : $x=y \Rightarrow x=4\pm\sqrt{6}$

$\therefore \quad OD=\sqrt{2}\left(4-\sqrt{6}\right)=4\sqrt{2}-2\sqrt{3}$

9. (2) : $BD=\dfrac{ac}{b+c}$, $\dfrac{AI}{ID}=\dfrac{b+c}{a}$

$\dfrac{AI}{ID}.\dfrac{BI}{IE}.\dfrac{CI}{IP}-\Sigma\dfrac{AI}{ID}=$

$\dfrac{(b+c)(c+a)(a+b)-bc(b+c)-ca(c+a)-ab(a+b)}{abc}$

$=\dfrac{2abc}{abc}=2$

10. (b) : **P.** $23^2=-1 \bmod 53$, $23^{22}=-1 \bmod 53$, $23^{23}=-23 \bmod 53=30 \bmod 53$

Q. $X=x+3$, $Y=y+2$, $Z=z+1$, $U=u$

$\Rightarrow \quad X+Y+Z+U=9$

Number of solutions $=\binom{8}{3}=56$

R. $\sum\dfrac{5!}{\alpha!\beta!\gamma!}1^{\alpha}.x^{\beta}.(zy)^{\gamma}$, $\beta=2$, $\gamma=1$, $\alpha=2$

Coefficient $=\dfrac{5!}{2!\cdot 2!}.2=60$

S. $\sum_{r=1}^{10}r\dfrac{(11-r)}{r}=1+2+...+10=55$

Solution Sender of Maths Musing

SET-136

1. Khwaja Sami Baig : Azamgarh (UP)

SET-135

1. Satya dev : Bangalore

YOU ASKED WE ANSWERED

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Use the vast expertise of our mtg team to get to the bottom of the question. From the serious to the silly, the controversial to the trivial, the team will tackle the questions, easy and tough.

The best questions and their solutions will be printed in this column each month.

Q.1 The tangent at a point $P(a\cos\phi,\ b\sin\phi)$ of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, meets its auxiliary circle in two points, the chord joining which subtends a right angle at the centre, then find the eccentricity of the ellipse.

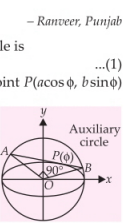
– Ranveer, Punjab

Ans. Equation of the auxiliary circle is

$x^2+y^2=a^2$ ...(1)

Equation of the tangent at point $P(a\cos\phi, b\sin\phi)$ is

$\left(\frac{x}{a}\right)\cos\phi+\left(\frac{y}{b}\right)\sin\phi=1$ ...(2)



which meets the auxiliary circle at points A and B.

$\therefore$ Equation of pair of lines OA and OB is obtained by making homogeneous (1) with the help of (2) as

$$x^2+y^2=a^2\left(\frac{x}{a}\cos\phi+\frac{y}{b}\sin\phi\right)^2$$

$$\Rightarrow\ (1-\cos^2\phi)x^2-\frac{2xya\sin\phi\cos\phi}{b}+\left(1-\frac{a^2}{b^2}\sin^2\phi\right)y^2=0$$

But $\angle AOB=90°$

$\therefore$ coefficient of x^2 + coefficient of $y^2=0$

$$\Rightarrow\quad 1-\cos^2\phi+1-\left(\frac{a^2}{b^2}\right)\sin^2\phi=0$$

$$\Rightarrow\quad \sin^2\phi\left(1-\frac{a^2}{b^2}\right)+1=0$$

$$\Rightarrow\quad (a^2-b^2)\sin^2\phi=b^2$$

$$\Rightarrow\quad a^2e^2\sin^2\phi=a^2(1-e^2)$$

$$\Rightarrow\quad (1+\sin^2\phi)\,e^2=1$$

$$\Rightarrow\quad e=\frac{1}{\sqrt{1+\sin^2\phi}}=(1+\sin^2\phi)^{-1/2}.$$

Q.2 If $f(x)=\sqrt{3|x|-x-2}$ and $g(x)=\sin x$, find the domain of definition of $fog(x)$.

– V. Rajesh, Kerala

Ans. We have, $f(x)=\sqrt{3|x|-x-2}$ and $g(x)=\sin x$

$\therefore\quad fog(x)=\sqrt{3|\sin x|-\sin x-2}$

which is defined, if $3|\sin x|-\sin x-2\geq 0$

If $\sin x>0$, then $2\sin x-2\geq 0\Rightarrow\sin x\geq 1$

$$\Rightarrow\quad \sin x=1\Rightarrow x=2n\pi+\frac{\pi}{2}.$$

If $\sin x<0$, then $-4\sin x-2\geq 0\Rightarrow -1\leq\sin x\leq -\frac{1}{2}$

$$\therefore\quad x\in\left\{2n\pi+\frac{7\pi}{6},\ 2n\pi+\frac{11\pi}{6}\right\}\cup\left\{2m\pi+\frac{\pi}{2}\right\},\ n,m\in I.$$

Q.3 For sequences $\{A_n\}$ and $\{B_n\}$ of events on some sample space S, let $\lim_{n\to\infty}P(B_n)=1$.

Determine $\lim_{n\to\infty}[P(A_n/B_n)-P(A_n)]$, where $P(A/B)$, is the conditional probability of A given B.

– Madhav, Assam

Ans. Since $\lim_{n\to\infty}P(B_n)=1$ ...(1)

$\lim_{n\to\infty}[1-P(B_n)]=0$ ...(2)

Further,

$P(A_n\cup B_n)\geq P(B_n)\Rightarrow\lim_{n\to\infty}P(A_n\cup B_n)=1$ ....(3)

Now, $P\left(\frac{A_n}{B_n}\right)-P(A_n)=\frac{P(A_n\cap B_n)}{P(B_n)}-P(A_n)$

$$=\frac{P(A_n\cap B_n)-P(A_n)\cdot P(B_n)}{P(B_n)}$$

$$=\frac{P(A_n)+P(B_n)-P(A_n\cup B_n)-P(A_n)\cdot P(B_n)}{P(B_n)}$$

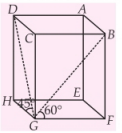
$$=\frac{P(A_n)[1-P(B_n)]+P(B_n)-P(A_n\cup B_n)}{P(B_n)}$$

$\therefore\quad \lim_{n\to\infty}\left[P\left(\frac{A_n}{B_n}\right)-P(A_n)\right]=$

$$\lim_{n\to\infty}\frac{\{(P(A_n))[1-P(B_n)]+P(B_n)-P(A_n\cup B_n)\}}{\lim_{n\to\infty}P(B_n)}$$

$$=\frac{0+1-1}{1}=0\ \text{[From equations 1, 2 and 3].}$$

Q.4 In the adjoining figure of a rectangular solid, it is given that $\angle DGH = 45°$ and $\angle BGF = 60°$.



Then $\cos(\angle BGD) =$

– Deepak, Rajasthan

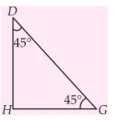
Ans. $\angle DGH = 45°$ and $\angle BGF = 60°$.

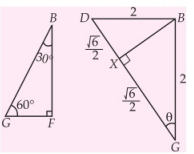
Let $GF = EH = BC = AD = 1$ unit

$BF = CG = DH = AE = \sqrt{3}$ units

$HG = EF = AB = CD = \sqrt{3}$ units

Draw altitude BX.



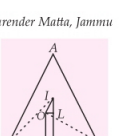


$$DX = XG = \frac{\sqrt{6}}{2}, \quad \cos\angle BGD = \frac{XG}{BG} = \frac{\sqrt{6}}{4}.$$

Q.5 The line joining the incentre to the circumcentre of a triangle ABC is inclined to the side BC at which angle?

– Surender Matta, Jammu

Ans. Let I be the in centre and O be the circum-centre of the triangle ABC. Let OL be parallel to BC. Let $\angle IOL = \theta$. $IM = r$, $OC = R$, $\angle NOC = A$.



$$\Rightarrow \quad \tan\theta = \frac{IL}{OL} = \frac{IM - LM}{BM - BN} = \frac{IM - ON}{BM - NC}$$

$$= \frac{r - R\cos A}{r\cot\frac{B}{2} - R\sin A}$$

$$= \frac{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} - R\cos A}{4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\cdot\cot\frac{B}{2} - R\sin A}$$

$$= \frac{\cos A + \cos B + \cos C - 1 - \cos A}{\sin A + \sin C - \sin B - \sin A}$$

$$= \frac{\cos B + \cos C - 1}{\sin C - \sin B}$$

$$\Rightarrow \quad \theta = \tan^{-1}\left(\frac{\cos B + \cos C - 1}{\sin C - \sin B}\right)$$

Q.6 If a, b and d are lengths of a side, a shortest diagonal and a longest diagonal of a regular nonagon, then prove that $a + b = d$.

– Shekhar, Trivandrum

Ans. The chords of lengths a, b, d Subtend central angles 40°, 80° and 160° respectively.

$a = 2r\sin 20°$, $b = 2r\sin 40°$, $d = 2r\sin 80°$

where r is the radius of the circle circumscribing the nonagon $\sin 40° + \sin 20° = 2\sin 30°\cos 10° = \cos 10° = \sin 80°$

$\Rightarrow$ $2r\sin 40° + 2r\sin 20° = 2r\sin 80°$

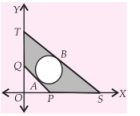
$\Rightarrow$ $a + b = d$.

Q.7 In the XY plane, two points $A(2, 2)$ and $B(7, 7)$ are taken. R is the region in the first quadrant which consists of points C such that triangle ABC is an acute angled triangle. Find the closest integer to the area of the region R.

– Mansi Goel, Patna

Ans. For ΔABC, all the angles of the triangle must be acute. For A to be acute, C must lie above the line passing through A and $\perp$ to AB. The segment of that line in the first quadrant lies between $P(4, 0)$ and $Q(0, 4)$. For the angle to be acute, the point C must lie below the line through B and $\perp$ to AB. *i.e.*, the segment of that line in the first quadrant lies between $S(14, 0)$ and $T(0, 14)$

For C to be acute, point C must lie outside the circle of diameter AB.



Let O be the origin.

The region is shaded whose area

$= \Delta OST - \Delta OPQ$ – circle area

$$= \frac{1}{2}14^2 - \frac{1}{2}4^2 - \pi\left(\frac{\sqrt{50}}{2}\right)^2$$

$$= 90 - \frac{25}{2}\pi \approx 51$$

■■

MIND BENDERS

Q.1 While reading a newspaper, you notice that four pages of one section are missing. One of the missing pages is page 5. The back page of the newspaper is page 24. What are the other three missing pages ?

Ans. Newspaper are printed double-sided, two pages to a sheet. The first and second pages are attached to the second-to last and last pages - in this case, pages 23 and 24. The rest of the pages are attached as follows :

1 - 2 with 23 - 24
3 - 4 with 21 - 22
5 - 6 with 19 - 20
7 - 8 with 17 - 18
9 - 10 with 15 - 16
11 - 12 with 13 - 14

So, pages 6, 19 and 20 are also missing.

Q.2 A school teacher uses a five-hour hourglass to keep track of class time. One day, he sets the hourglass at 9 a.m. and while he is teaching his class, a student inadvertently inverts the hourglass. Another student who notices this, sets the hourglass to its initial position at 11:30 a.m. In this way, the class ends at 3 p.m. At what time did the first student invert the hourglass?

Ans. After being inverted twice, the hourglass continued working in its initial position. Therefore, the extra hour that it measured was a consequence of these two inversions, half an hour each time. If it was inverted for the second time at 11 : 30 a.m., the first time had to be a half-hour earlier, at 11 : 00 a.m.

Q.3 A shepherd says to another, "If I give you one sheep, you will have twice the number of sheep that I have, but if you give me one, we will both have the same number of sheep." How many sheep did each shepherd have?

Ans. The shepherd that is talking had 5 sheep and the other one had 7.

Q.4 You have three cards : an ace, a queen and a six. One is a diamond, one is a heart, and one is spade, although not necessarily in that order.

- The diamond sits between the queen and the heart.
- The six is immediately to the right of the spade.

So, where will be each card located?

Ans. From left to right : queen of spades, six of diamonds and ace of hearts.

Q.5 A bookstore has a nice discount policy. If you buy a ₹ 20 book today, you get a 2% discount on your next purchase. If you buy a ₹ 15 book, you get a 1.5% discount on your next purchase. If you have to buy three books that costs ₹ 10, ₹ 20 and ₹ 30, you could buy the ₹ 30 book today, the ₹ 10 tomorrow (on which you will get a 3% discount), and the ₹ 20 book the following day (on which you will get a 1% discount). Or you could buy the ₹ 30 book and the ₹ 20 book today, and the ₹ 10 book tomorrow (with a 5% discount). What is the cheapest way to buy five books priced at ₹ 10, ₹ 20, ₹ 30, ₹ 40 and ₹ 50?

Ans. You first buy books for ₹ 80 and the next day, for ₹ 70, which represents a discount of ₹ 70 × 0.08 = ₹ 5.60. (It will be the same result by inverting the order of the purchases - first the ₹ 70 purchase and the next day the ₹ 80 one)

Q.6 Akash and Tanuj played several golf matches against each other in a week. They played for a pizza at each match, but no pizzas were purchased until the end of the week. If at any time during the week Tanuj and Akash had the same number of wins, those pizzas were cancelled. Akash won four matches (but no pizzas) and Tanuj won three pizzas. How many rounds of golf were played?

Ans. Tanuj had to win four matches to draw even with Akash, and then Tanuj had to win three more times.

4 + 4 + 3 = 11

So, Eleven rounds of golf were played. ■■

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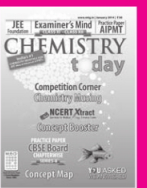

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